Pearson Edexcel

Mark Scheme (Results)

Summer 2019

Pearson Edexcel GCE Mathematics

A Further Pure Maths 2 (9FM0/02)

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General Marking Guidance

- All candidates must receive the same treatment. Examiners must mark the first candidate in exactly the same way as they mark the last.
- Mark schemes should be applied positively. Candidates must be rewarded for what they have shown they can do rather than penalised for omissions.
- Examiners should mark according to the mark scheme not according to their perception of where the grade boundaries may lie.
- There is no ceiling on achievement. All marks on the mark scheme should be used appropriately.
- All the marks on the mark scheme are designed to be awarded. Examiners should always award full marks if deserved, i.e. if the answer matches the mark scheme. Examiners should also be prepared to award zero marks if the candidate's response is not worthy of credit according to the mark scheme.
- Where some judgement is required, mark schemes will provide the principles by which marks will be awarded and exemplification may be limited.
- When examiners are in doubt regarding the application of the mark scheme to a candidate's response, the team leader must be consulted.
- Crossed out work should be marked UNLESS the candidate has replaced it with an alternative response.

PEARSON EDEXCEL GCE MATHEMATICS

General Instructions for Marking

1. The total number of marks for the paper is 75

2. The Edexcel Mathematics mark schemes use the following types of marks:

- **M** marks: Method marks are awarded for 'knowing a method and attempting to apply it', unless otherwise indicated.
- **A** marks: Accuracy marks can only be awarded if the relevant method (M) marks have been earned.
- **B** marks are unconditional accuracy marks (independent of M marks)
- Marks should not be subdivided.

3. Abbreviations

   These are some of the traditional marking abbreviations that will appear in the mark schemes.

   - bod – benefit of doubt
   - ft – follow through
   - the symbol $\sqrt{}$ will be used for correct ft
   - cao – correct answer only
   - cso - correct solution only. There must be no errors in this part of the question to obtain this mark
   - isw – ignore subsequent working
   - awrt – answers which round to
   - SC: special case
   - oe – or equivalent (and appropriate)
   - d… or dep – dependent
   - indep – independent
   - dp decimal places
   - sf significant figures
   - * The answer is printed on the paper or ag- answer given
   - □ or d… The second mark is dependent on gaining the first mark

4. All A marks are 'correct answer only' (cao.), unless shown, for example, as A1 ft to indicate that previous wrong working is to be followed

through. After a misread however, the subsequent A marks affected are treated as A ft, but manifestly absurd answers should never be awarded A marks.

5. For misreading which does not alter the character of a question or materially simplify it, deduct two from any A or B marks gained, in that part of the question affected.

6. If a candidate makes more than one attempt at any question:
   - If all but one attempt is crossed out, mark the attempt which is NOT crossed out.
   - If either all attempts are crossed out or none are crossed out, mark all the attempts and score the highest single attempt.

7. Ignore wrong working or incorrect statements following a correct answer.

General Principles for Core Mathematics Marking

(But note that specific mark schemes may sometimes override these general principles).

Method mark for solving 3 term quadratic:

1. Factorisation

$(x^2 + bx + c) = (x + p)(x + q)$, where $|pq| = |c|$, leading to x = …

$(ax^2 + bx + c) = (mx + p)(nx + q)$, where $|pq| = |c|$ and $|mn| = |a|$, leading to x = …

2. Formula

Attempt to use the correct formula (with values for a, b and c).

3. Completing the square

Solving $x^2 + bx + c = 0$: $\left(x \pm \frac{b}{2}\right)^2 \pm q \pm c = 0, \quad q \neq 0$, leading to x = …

Method marks for differentiation and integration:

1. Differentiation

Power of at least one term decreased by 1. ($x^n \to x^{n-1}$)

2. Integration

Power of at least one term increased by 1. ($x^n \to x^{n+1}$)

Use of a formula

Where a method involves using a formula that has been learnt, the advice given in recent examiners’ reports is that the formula should be quoted first.

Normal marking procedure is as follows:

Method mark for quoting a correct formula and attempting to use it, even if there are small errors in the substitution of values.

Where the formula is not quoted, the method mark can be gained by implication from correct working with values, but may be lost if there is any mistake in the working.

Exact answers

Examiners’ reports have emphasised that where, for example, an exact answer is asked for, or working with surds is clearly required, marks will normally be lost if the candidate resorts to using rounded decimals.

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Mark Scheme – Pre-Stand

Question	Scheme	Marks	AOs
1(a)	$y=\tanh^{-1}(x)\Rightarrow \tanh y = x \Rightarrow x=\dfrac{\sinh y}{\cosh y}=\dfrac{e^y-e^{-y}}{e^y+e^{-y}}$	M1 A1	2.1 1.1b
	Note that some candidates only have one variable and reach e.g. $x=\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$ or $\tanh x=\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$ Allow this to score M1A1		
	$x\left(e^{2y}+1\right)=e^{2y}-1\Rightarrow e^{2y}(1-x)=1+x\Rightarrow e^{2y}=\dfrac{1+x}{1-x}$	M1	1.1b
	$e^{2y}=\dfrac{1+x}{1-x}\Rightarrow 2y=\ln\left(\dfrac{1+x}{1-x}\right)\Rightarrow y=\dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)*$	A1*	2.1
	Note that $e^{2y}(x-1)+x+1=0$ can be solved as a quadratic in e^y: $e^y=\dfrac{-\sqrt{0-4(x-1)(x+1)}}{2(x-1)}=\dfrac{-\sqrt{4(1-x)(x+1)}}{2(x-1)}=\dfrac{2\sqrt{(1-x)(x+1)}}{2(1-x)}$ $=\dfrac{\sqrt{(x+1)}}{\sqrt{(1-x)}}\Rightarrow y=\dfrac{1}{2}\ln\dfrac{(x+1)}{(1-x)}*$ Score **M1** for an attempt at the quadratic formula to make e^y the subject (condone ± √…) and **A1*** for a correct solution that rejects the positive root at some point and deals with the $(x-1)$ bracket correctly		
	$k = 1$ or $-1 < x < 1$	B1	1.1b
		(5)	
(a) **Way 2**	$\tanh^{-1}x=\dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)\Rightarrow x=\tanh\left(\dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)\right)=\dfrac{e^{\ln\frac{1+x}{1-x}}-1}{e^{\ln\frac{1+x}{1-x}}+1}$	M1 A1	2.1 1.1b
	$x=\dfrac{e^{\ln\frac{1+x}{1-x}}-1}{e^{\ln\frac{1+x}{1-x}}+1}=\dfrac{\frac{1+x}{1-x}-1}{\frac{1+x}{1-x}+1}=x$ Hence true, QED, tick etc.	M1 A1	1.1b 2.1
(b)	$2x=\tanh\left(\ln\sqrt{2-3x}\right)\Rightarrow \tanh^{-1}(2x)=\ln\sqrt{2-3x}$	M1	3.1a
	$\dfrac{1}{2}\ln\left(\dfrac{1+2x}{1-2x}\right)=\dfrac{1}{2}\ln(2-3x)\Rightarrow\dfrac{1+2x}{1-2x}=2-3x$	M1	2.1
	$6x^2-9x+1=0$	A1	1.1b
	$6x^2-9x+1=0\Rightarrow x=...$	M1	1.1b
	$x=\dfrac{9-\sqrt{57}}{12}$	A1	3.2a
		(5)	

	Alternative for first 2 marks of (b)		
	$2x=\tanh\left(\ln\sqrt{2-3x}\right)\Rightarrow 2x=\dfrac{e^{2\ln\sqrt{2-3x}}-1}{e^{2\ln\sqrt{2-3x}}+1}$	M1	3.1a
	$\Rightarrow\dfrac{2-3x-1}{2-3x+1}=2x$	M1	2.1
			(10 marks)

Notes

(a)

<u>If you come across any attempts to use calculus to prove the result – send to review</u>

M1: Begins the proof by expressing tanh in terms of exponentials and forms an equation in exponentials.

The exponential form can be any of $\dfrac{\left(e^y-e^{-y}\right)/2}{\left(e^y+e^{-y}\right)/2}$, $\dfrac{e^y-e^{-y}}{e^y+e^{-y}}$, $\dfrac{e^{2y}-1}{e^{2y}+1}$

Allow any variables to be used **but the final answer must be in terms of *x***. Allow alternative notation for $\tanh^{-1}x$ e.g. artanh, arctanh.

A1: Correct expression for "x" in terms of exponentials

M1: Full method to make $e^{2"y"}$ the subject of the formula. This must be correct algebra so allow sign errors only.

A1*: Completes the proof by using logs correctly and reaches the printed answer with no errors. Allow e.g. $\frac{1}{2}\ln\left(\frac{x+1}{1-x}\right)$, $\frac{1}{2}\ln\frac{x+1}{1-x}$, $\frac{1}{2}\ln\left|\frac{x+1}{1-x}\right|$. Need to see $\tanh^{-1}x=\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ as a conclusion but allow if the proof concludes that $y=\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ with y defined as $\tanh^{-1}x$ earlier.

B1: Correct value for k or writes $-1 < x < 1$

Way 2

M1: Starts with result, takes tanh of both sides and expresses in terms of exponentials

A1: Correct expression

M1: Eliminates exponentials and logs and simplifies

A1: Correct result (i.e. $x = x$) with conclusion

B1: Correct value for k or writes $-1 < x < 1$

(b)

M1: Adopts a correct strategy by taking $\tanh^{-1}$ of both sides

M1: Makes the link with part (a) by replacing artanh($2x$) with $\frac{1}{2}\ln\left(\frac{1+2x}{1-2x}\right)$ and demonstrates the use of the power law of logs to obtain an equation with logs removed **correctly**.

A1: Obtains the correct 3TQ

M1: Solves their 3TQ using a correct method (see General Guidance – if no working is shown (calculator) and the roots are correct for their quadratic, allow M1)

A1: Correct value with the other solution rejected (accept rejection by omission) so $x=\dfrac{9\pm\sqrt{57}}{12}$ scores A0 unless the positive root is rejected

Alternative for first 2 marks of (b)

M1: Adopts a correct strategy by expressing tanh in terms of exponentials

M1: Demonstrates the use of the power law of logs to obtain an equation with logs removed correctly

Question	Scheme	Marks	AOs
2(i)	$p+q+r=2, \quad pq+pr+qr=4, \quad pqr=5$	B1	3.1a
	$\frac{2}{p}+\frac{2}{q}+\frac{2}{r}=\frac{2(pq+pr+qr)}{pqr}$	M1	1.1b
	$=\frac{8}{5}$	A1ft	1.1b
		(3)	
	Alternative for part (i)		
	$x=\frac{2}{y} \Rightarrow \frac{8}{y^3}-\frac{8}{y^2}+\frac{8}{y}-5=0 \Rightarrow 5y^3-8y^2+8y-8=0$	B1	3.1a
	$\frac{2}{p}+\frac{2}{q}+\frac{2}{r}=-\frac{-8}{5}$	M1	1.1b
	$=\frac{8}{5}$	A1ft	1.1b
		(3)	
(ii)	$(p-4)(q-4)(r-4)=(pq-4p-4q+16)(r-4)$ $=pqr-4pq-4pr-4qr+16p+16q+16r-64$	M1 A1	1.1b 1.1b
	$\left(=pqr-4(pq+pr+qr)+16(p+q+r)-64\right)$		
	$=5-4(4)+16(2)-64=-43$	A1	1.1b
		(3)	
	Alternative for part (ii)		
	$(x+4)^3-2(x+4)^2+4(x+4)-5=0$	M1	1.1b
	$=...64+...-32+...16+...-5=43$	A1	1.1b
	$\therefore (p-4)(q-4)(r-4)=-43$	A1	1.1b
		(3)	
(iii)	E.g. $p^3+q^3+r^3=$ $=(p+q+r)^3-3(p+q+r)(pq+pr+qr)+3pqr$ or $=(p+q+r)\left((p+q+r)^2-2(pq+pr+qr)-pq-pr-qr\right)+3pqr$ or $=2\left((p+q+r)^2-2(pq+pr+qr)\right)-4(p+q+r)+3pqr$ $\Rightarrow p^3+q^3+r^3=...$	M1	3.1a
	$=2^3-3(2)(4)+3(5)=-1$ $=2\left(2^2-3(4)\right)+3(5)=-1$ $=2\left(2^2-2(4)\right)-4(2)+3(5)=-1$	A1	1.1b
		(2)	

	Alternative for part (iii)		
	$p^3-2p^2+4p-5=0,\ q^3-2q^2+4q-5=0,\ r^3-2r^2+4r-5=0$ $p^3+q^3+r^3-2\left(p^2+q^2+r^2\right)+4\left(p+q+r\right)-15=0$ $p^3+q^3+r^3=2\left(\left(p+q+r\right)^2-2\left(pq+pr+qr\right)\right)-4\left(p+q+r\right)+15$ $\Rightarrow p^3+q^3+r^3=...$	M1	3.1a
	$=2\left(2^2-2\left(4\right)\right)-4\left(2\right)+15=-1$	A1	1.1b
		(2)	
			(8 marks)

Notes

(i)
B1: Identifies the correct values for all 3 expressions (can score anywhere). Allow notation such as $\sum p,\ \sum pq$ for the sum and pair sum.
M1: Uses a correct identity for the sum
A1ft: Correct value (follow through their 2, 4 and 5)
Alternative:
B1: Obtains the correct cubic in "y"
M1: Uses a correct method
A1ft: Correct value (follow through their 2, 4 and 5)
(ii)
M1: Attempt to expand – must have an expression that involves the sum, pair sum and product
A1: Correct expansion
A1: Correct value
Alternative:
M1: Substitutes $x + 4$ for x in the given cubic
A1: Calculates the correct constant term
A1: Correct value
(iii)
M1: Establishes a correct identity that is in terms of the sum, pair sum and product and substitutes to reach a numerical expression for $p^3+q^3+r^3$
A1: Correct value

Question	Scheme	Marks	AOs
3(a) Way 1	$x = \frac{3}{2}\sinh u$	B1	2.1
	$\int \frac{dx}{\sqrt{4x^2+9}} = \int \frac{1}{\sqrt{4\left(\frac{9}{4}\right)\sinh^2 u + 9}} \times \frac{3}{2}\cosh u \, du$	M1	3.1a
	$= \int \frac{1}{2} \, du$	A1	1.1b
	$= \int \frac{1}{2} \, du = \frac{1}{2}u = \frac{1}{2}\sinh^{-1}\left(\frac{2x}{3}\right) + c$	A1	1.1b
		(4)	
(a) Way 2	$x = \frac{3}{2}\tan u$	B1	2.1
	$\int \frac{dx}{\sqrt{4x^2+9}} = \int \frac{1}{\sqrt{4\left(\frac{9}{4}\right)\tan^2 u + 9}} \times \frac{3}{2}\sec^2 u \, du$	M1	3.1a
	$= \int \frac{1}{2}\sec u \, du$	A1	1.1b
	$= \frac{1}{2}\ln\left(\sec u + \tan u\right) = \frac{1}{2}\ln\left(\frac{2x}{3} + \sqrt{1 + \left(\frac{2x}{3}\right)^2}\right)$ $u = \frac{1}{2}\sinh^{-1}\left(\frac{2x}{3}\right) + c$	A1	1.1b
(a) Way 3	$x = \frac{1}{2}u$ or $x = ku$ where $k > 0$ $k \neq 1$	B1	2.1
	$\int \frac{dx}{\sqrt{4x^2+9}} = \int \frac{1}{\sqrt{4\left(\frac{1}{4}\right)u^2 + 9}} \times \frac{1}{2} du$	M1	3.1a
	$= \frac{1}{2}\int \frac{1}{\sqrt{u^2+9}} du \quad \left(\text{or } \frac{1}{2}\int \frac{1}{\sqrt{u^2 + \frac{9}{4k^2}}} du \text{ for } x = ku\right)$	A1	1.1b
	$= \frac{1}{2}\sinh^{-1}\frac{u}{3} = \frac{1}{2}\sinh^{-1}\frac{2x}{3} + c$	A1	1.1b
(b)	Mean value = $\frac{1}{3(-0)}\left[\frac{1}{2}\sinh^{-1}\left(\frac{2x}{3}\right)\right]_0^3 = \frac{1}{3} \times \frac{1}{2}\sinh^{-1}\left(\frac{2 \times 3}{3}\right)(-0)$	M1	2.1
	$= \frac{1}{6}\ln\left(2 + \sqrt{5}\right)$ **(Brackets are required)**	A1ft	1.1b
		(2)	
			(6 marks)

Notes
(a) B1: Selects an appropriate substitution leading to an integrable form M1: Demonstrates a fully correct method for the substitution that includes substituting into the function and dealing with the "dx". The substitution being substituted does not need to be "correct" for this mark but the substitution must be an attempt at $\int \frac{1}{\sqrt{4\left[\mathrm{f}(u)\right]^2+9}} \times \mathrm{f}'(u)\,\mathrm{d}u$ with the $\mathrm{f}'(u)$ correct for their substitution. E.g. if $x = \frac{1}{2}u$ is used, must see $\mathrm{d}x = \frac{1}{2}\mathrm{d}u$ not 2du. A1: Correct simplified integral in terms of u from correct work and from a correct substitution A1: Correct answer including "+ c". Allow arcsinh or arsinh for $\sinh^{-1}$ from correct work and from a correct substitution (b) M1: Correctly applies the method for the mean value for their integration which must be of the form specified in part (a) and substitutes the limits 0 and 3 but condone omission of 0 A1: Correct exact answer (follow through their A and B). **Brackets are required if appropriate.**

Question	Scheme	Marks	AOs
4(a) Way 1	$C+iS=\cos\theta+i\sin\theta+\frac{1}{2}(\cos5\theta+i\sin5\theta)\left(+\frac{1}{4}(\cos9\theta+i\sin9\theta)+...\right)$	M1	1.1b
	$=e^{i\theta}+\frac{1}{2}e^{5i\theta}\left(+\frac{1}{4}e^{9i\theta}+...\right)$	A1	2.1
	$C+iS=\frac{e^{i\theta}}{1-\frac{1}{2}e^{4i\theta}}$	M1	3.1a
	$=\frac{2e^{i\theta}}{2-e^{4i\theta}}$ *	A1*	1.1b
		(4)	
(a) Way 2	$C+iS=\cos\theta+i\sin\theta+\frac{1}{2}(\cos5\theta+i\sin5\theta)\left(+\frac{1}{4}(\cos9\theta+i\sin9\theta)+...\right)$	M1	1.1b
	$C+iS=\cos\theta+i\sin\theta+\frac{1}{2}(\cos\theta+i\sin\theta)^5\left(+\frac{1}{4}(\cos\theta+i\sin\theta)^9+...\right)$	A1	2.1
	$C+iS=\frac{\cos\theta+i\sin\theta}{1-\frac{1}{2}(\cos\theta+i\sin\theta)^4}=\frac{e^{i\theta}}{1-\frac{1}{2}e^{4i\theta}}$	M1	3.1a
	$=\frac{2e^{i\theta}}{2-e^{4i\theta}}$ *	A1*	1.1b
		(4)	
(b) Way 1	$\frac{2e^{i\theta}}{2-e^{4i\theta}}\times\frac{2-e^{-4i\theta}}{2-e^{-4i\theta}}$	M1	3.1a
	$\frac{4e^{i\theta}-2e^{-3i\theta}}{4-2e^{-4i\theta}-2e^{4i\theta}+1}$	A1	1.1b
	$\frac{4\cos\theta+4i\sin\theta-2\cos3\theta+2i\sin3\theta}{5-2\cos4\theta+2i\sin4\theta-2\cos4\theta-2i\sin4\theta}$ **Dependent on the first M**	**d**M1	2.1
	$S=\frac{4\sin\theta+2\sin3\theta}{5-4\cos4\theta}$ *	A1*	1.1b
		(4)	
(b) Way 2	$\frac{2e^{i\theta}}{2-e^{4i\theta}}=\frac{2(\cos\theta+i\sin\theta)}{2-(\cos4\theta+i\sin4\theta)}\times\frac{2-(\cos4\theta-i\sin4\theta)}{2-(\cos4\theta-i\sin4\theta)}$	M1	3.1a
	$\frac{4\cos\theta+4i\sin\theta-2\cos\theta\cos4\theta-2\sin\theta\sin4\theta+2i\sin4\theta\cos\theta-2i\sin\theta\cos4\theta}{4+\cos^2 4\theta+\sin^2 4\theta-4\cos4\theta}$	A1	1.1b
	$\frac{4\cos\theta+4i\sin\theta-2\cos3\theta+2i\sin3\theta}{5-2\cos4\theta+2i\sin4\theta-2\cos4\theta-2i\sin4\theta}$ **Dependent on the first M**	**d**M1	2.1
	$S=\frac{4\sin\theta+2\sin3\theta}{5-4\cos4\theta}$ *	A1*	1.1b
		(8 marks)	

Notes
(a) **Way 1** M1: Combines the two series by pairing the multiples of θ (At least up to 5θ) A1: Converts to Euler form correctly (At least up to 5θ) M1: Recognises that C + iS is a convergent geometric series and uses the sum to infinity of a GP A1*: Reaches the printed answer with no errors **Way 2** M1: Combines the two series by pairing the multiples of θ (At least up to 5θ) A1: Converts to power form correctly (At least up to 5θ) M1: Recognises that C + iS is a convergent geometric series and uses the sum to infinity of a GP A1*: Reaches the printed answer with no errors (b) **Way 1** M1: Multiplies numerator and denominator by $2-\mathrm{e}^{-4\mathrm{i}\theta}$ A1: Correct fraction in terms of exponentials **d**M1: Converts back to trigonometric form A1*: Reaches the printed answer with no errors **Way 2** M1: Converts back to trigonometric form and realises the need to make the denominator real and multiplies numerator and denominator by the complex conjugate of the denominator which is **correct** for their fraction A1: Correct fraction in terms of trigonometric functions **d**M1: Uses the correct addition formula to obtain sin 3θ in the numerator A1*: Reaches the printed answer with no errors

Question	Scheme	Marks	AOs
5(a)	$4m^2+4m+37=0 \Rightarrow m=-\frac{1}{2}\pm 3\text{i}$	M1	1.1b
	$h=\text{e}^{-0.5t}\left(A\cos 3t+B\sin 3t\right)$	A1	1.1b
		(2)	
(b)	$t=0,\ \ h=-20 \Rightarrow A=-20$	M1	3.4
	$\frac{\text{d}h}{\text{d}t}=-0.5\text{e}^{-0.5t}\left(A\cos 3t+B\sin 3t\right)+\text{e}^{-0.5t}\left(-3A\sin 3t+3B\cos 3t\right)$ $t=0,\ \frac{\text{d}h}{\text{d}t}=55 \Rightarrow B=...$(NB B = 15)	M1	3.4
	$\left(h=\right)\text{e}^{-0.5t}\left(15\sin 3t-20\cos 3t\right)$	A1	1.1b
	$-0.5\text{e}^{-0.5t}\left(15\sin 3t-20\cos 3t\right)+\text{e}^{-0.5t}\left(60\sin 3t+45\cos 3t\right)=0$ or e.g. $-0.5\text{e}^{-0.5t}\left(15\sin 3t-20\cos 3t\right)+\frac{25\sqrt{37}}{2}\text{e}^{-0.5t}\sin\left(3t+\arctan\frac{22}{21}\right)=0$ $\Rightarrow t=...$	M1	3.1b
	$\tan 3t=-\frac{22}{21}$ or e.g. $3t+\tan^{-1}\frac{22}{21}=0$	A1 **M1 on ePEN**	2.1
	t = 0.778 s	A1	1.1b
	$h=\text{e}^{-0.5\times"0.778"}\left(15\sin\left(3\times"0.778"\right)-20\cos\left(3\times"0.778"\right)\right)$	**d**M1	1.1b
	= 16.7 cm	A1	3.2a
		(8)	
(c)	E.g. considers large values of t in the model for h **or** states that for large values of t, h becomes smaller or becomes zero	M1	3.4
	E.g. • The value of h is very small when t is large and this is likely to be correct (as the displacement of end of the board should get smaller and smaller) • This suggests the model is suitable • This is realistic • This is suitable as the board will tend towards its equilibrium position • When t is large the value of h is never zero so the model is not really appropriate for large values of t	A1 **B1 on ePEN**	3.2b
		(2)	
		(12 marks)	

Notes

(a)

M1: Uses the model to form and solve the auxiliary equation $4m^2+4m+37=0$

See General Guidance for awarding this mark. This can be implied by correct values for m (from calculator)

A1: Correct general solution including "h ="

(b)

M1: Uses the model and the initial conditions to establish the value of "*A*". Need to see $t = 0$ and $h = \pm 20$ leading to a value for "*A*". This may be implied by $A = -20$ or $A = 20$.
M1: Differentiates their model using the product rule and uses the initial conditions, $t = 0$ with $dh/dt = \pm 55$, to establish the value of "*B*"
A1: Correct particular solution or correct values for *A* and *B*
M1: Uses their solution to the model with a correct strategy to obtain a value for *t* e.g. differentiates or uses their derivative from earlier, sets equal to zero and solves for *t*
A1(M1 on ePEN): Correct equation for *t*
A1: Correct value for *t* (allow awrt 0.778 if necessary) but this value may be implied.
dM1: Uses the model and their **positive** value for *t* to find the maximum displacement - **if their *t* is incorrect there must be some indication that they are using their *h* and not just a number written down. E.g. must see substitution into their *h* or they re-state their *h* and obtain a value for *h*.**
Dependent on all the previous method marks
A1: Correct value (awrt 16.7 (units not needed))
(c)
M1: Considers the model for large values of *t* either by substituting values or by considering the expression and commenting on its behaviour for large values of *t*. E.g. as $t \to \infty,\ h \to 0$ or as $t \to \infty,\ e^{-0.5t} \to 0$ or as $t \to \infty$ the oscillations become smaller etc.
A1: Makes a suitable comment – see scheme for examples

Question	Scheme	Marks	AOs
6(a)	Examples: $\begin{pmatrix}\cos 120 & -\sin 120\\ \sin 120 & \cos 120\end{pmatrix}\begin{pmatrix}6\\2\end{pmatrix}=...$ **or** $(6+2\text{i})\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}\text{i}\right)$ **or** $\sqrt{40}\left(\cos\arctan\left(\frac{2}{6}\right)+\text{i}\sin\arctan\left(\frac{2}{6}\right)\right)\left(\cos\left(\frac{2\pi}{3}\right)+\text{i}\sin\left(\frac{2\pi}{3}\right)\right)$ **or** $\sqrt{40}\left(\cos\left(\arctan\left(\frac{2}{6}\right)+\frac{2\pi}{3}\right)+\text{i}\sin\left(\arctan\left(\frac{2}{6}\right)+\frac{2\pi}{3}\right)\right)$ **or** $\sqrt{40}\text{e}^{\text{i}\arctan\left(\frac{2}{6}\right)}\text{e}^{\text{i}\left(\frac{2\pi}{3}\right)}$	M1	3.1a
	$\left(-3-\sqrt{3}\right)$ **or** $\left(3\sqrt{3}-1\right)\text{i}$	A1	1.1b
	$\left(-3-\sqrt{3}\right)+\left(3\sqrt{3}-1\right)\text{i}$	A1	1.1b
	Examples: $\begin{pmatrix}\cos 240 & -\sin 240\\ \sin 240 & \cos 240\end{pmatrix}\begin{pmatrix}6\\2\end{pmatrix}=...$ **or** $(6+2\text{i})\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}\text{i}\right)$ **or** $\sqrt{40}\left(\cos\arctan\left(\frac{2}{6}\right)+\text{i}\sin\arctan\left(\frac{2}{6}\right)\right)\left(\cos\left(\frac{4\pi}{3}\right)+\text{i}\sin\left(\frac{4\pi}{3}\right)\right)$ **or** $\sqrt{40}\left(\cos\left(\arctan\left(\frac{2}{6}\right)+\frac{4\pi}{3}\right)+\text{i}\sin\left(\arctan\left(\frac{2}{6}\right)+\frac{4\pi}{3}\right)\right)$ **or** $\sqrt{40}\text{e}^{\text{i}\arctan\left(\frac{2}{6}\right)}\text{e}^{\text{i}\left(\frac{4\pi}{3}\right)}$	M1	3.1a
	$\left(-3+\sqrt{3}\right)$ **or** $\left(-3\sqrt{3}-1\right)\text{i}$	A1	1.1b
	$\left(-3+\sqrt{3}\right)+\left(-3\sqrt{3}-1\right)\text{i}$	A1	1.1b
		(6)	
(b) Way 1	Area ABC = $3\times\frac{1}{2}\sqrt{6^2+2^2}\sqrt{6^2+2^2}\sin 120°$ or Area AOB = $\frac{1}{2}\sqrt{6^2+2^2}\sqrt{6^2+2^2}\sin 120°$	M1	2.1
	Area DEF = $\frac{1}{4}ABC$ or $\frac{3}{4}AOB$	**d**M1	3.1a
	$=\frac{3}{8}\times 40\times\frac{\sqrt{3}}{2}=\frac{15\sqrt{3}}{2}$	A1	1.1b
		(3)	

(b) Way 2	$D\left(\frac{3-\sqrt{3}}{2},\frac{3\sqrt{3}+1}{2}\right)$ $OD=\sqrt{\left(\frac{3-\sqrt{3}}{2}\right)^2+\left(\frac{3\sqrt{3}+1}{2}\right)^2}=\sqrt{10}$ Area DOF = $\frac{1}{2}\sqrt{10}\sqrt{10}\sin 120°$	M1	2.1
	Area DEF = $3DOF$	**d**M1	3.1a
	$=3\times\frac{1}{2}\times\sqrt{10}\sqrt{10}\times\frac{\sqrt{3}}{2}=\frac{15\sqrt{3}}{2}$	A1	1.1b
(b) Way 3	$AB=\sqrt{\left(9+\sqrt{3}\right)^2+\left(3-3\sqrt{3}\right)^2}=\sqrt{120}$ Area ABC = $\frac{1}{2}\sqrt{120}\sqrt{120}\sin 60°\left(=30\sqrt{3}\right)$	M1	2.1
	Area DEF = $\frac{1}{4}ABC$	**d**M1	3.1a
	$=\frac{1}{4}\times 30\sqrt{3}=\frac{15\sqrt{3}}{2}$	A1	1.1b
(b) Way 4	$D\left(\frac{3-\sqrt{3}}{2},\frac{3\sqrt{3}+1}{2}\right),\ E(-3,-1),\ F\left(\frac{3+\sqrt{3}}{2},\frac{-3\sqrt{3}+1}{2}\right)$ $DE=\sqrt{\left(\frac{3-\sqrt{3}}{2}+3\right)^2+\left(\frac{3\sqrt{3}+1}{2}+1\right)^2}\left(=\sqrt{30}\right)$ Area DEF = $\frac{1}{2}\sqrt{30}\sqrt{30}\sin 60°$	M1 **d**M1	2.1 3.1a
	$=\frac{15\sqrt{3}}{2}$	A1	1.1b
(b) Way 5	Area ABC = $\frac{1}{2}\begin{vmatrix}6 & -3-\sqrt{3} & \sqrt{3}-3 & 6\\ 2 & 3\sqrt{3}-1 & -3\sqrt{3}-1 & 2\end{vmatrix}=30\sqrt{3}$	M1	2.1
	Area DEF = $\frac{1}{4}ABC$	**d**M1	3.1a
	$=\frac{1}{4}\times 30\sqrt{3}=\frac{15\sqrt{3}}{2}$	A1	1.1b

(9 marks)

Notes

(a)
M1: Identifies a suitable method to rotate the given point by 120° (or equivalent) about the origin. May see equivalent work with modulus/argument or exponential form e.g. an attempt to multiply by $\cos\frac{2\pi}{3}+\mathrm{i}\sin\frac{2\pi}{3}$ or $\mathrm{e}^{\frac{2\pi}{3}\mathrm{i}}$
A1: Correct real part or correct imaginary part
A1: Completely correct complex number
M1: Identifies a suitable method to rotate the given point by 240° (or equivalent e.g. rotate their *B* by 120°) about the origin

May see equivalent work with modulus/argument or exponential form e.g. an attempt to multiply $6 + 2\text{i}$ by $\cos\frac{4\pi}{3} + \text{i}\sin\frac{4\pi}{3}$ or $\text{e}^{\frac{4\pi}{3}\text{i}}$ or their *B* by $\cos\frac{2\pi}{3} + \text{i}\sin\frac{2\pi}{3}$ or $\text{e}^{\frac{2\pi}{3}\text{i}}$

A1: Correct real part or correct imaginary part

A1: Completely correct complex number

(b)

In general, the marks in (b) should be awarded as follows:

M1: Attempts to find the area of a relevant triangle

dM1: completes the problem by multiplying by an appropriate factor to find the area of *DEF*

Dependent on the first method mark

A1: Correct exact area

In some cases it may not be possible to distinguish the 2 method marks. In such cases they can be awarded together for a direct method that finds the area of *DEF*

Examples:

Way 1

M1: A correct strategy for the area of a relevant triangle such as *ABC* or *AOB*

dM1: Completes the problem by linking the area of *DEF* correctly with *ABC* or with *AOB*

A1: Correct value

Way 2

M1: A correct strategy for the area of a relevant triangle such as *DOF*

dM1: Completes the problem by linking the area of *DEF* correctly with *DOF*

A1: Correct value

Way 3

M1: A correct strategy for the area of a relevant triangle such as *ABC*

dM1: Completes the problem by linking the area of *DEF* correctly with *ABC*

A1: Correct value

Way 4

M1**d**M1: A correct strategy for the area of *DEF*. Finds 2 midpoints and attempts one side of *DEF* and uses a correct triangle area formula. By implication this scores both M marks.

A1: Correct value

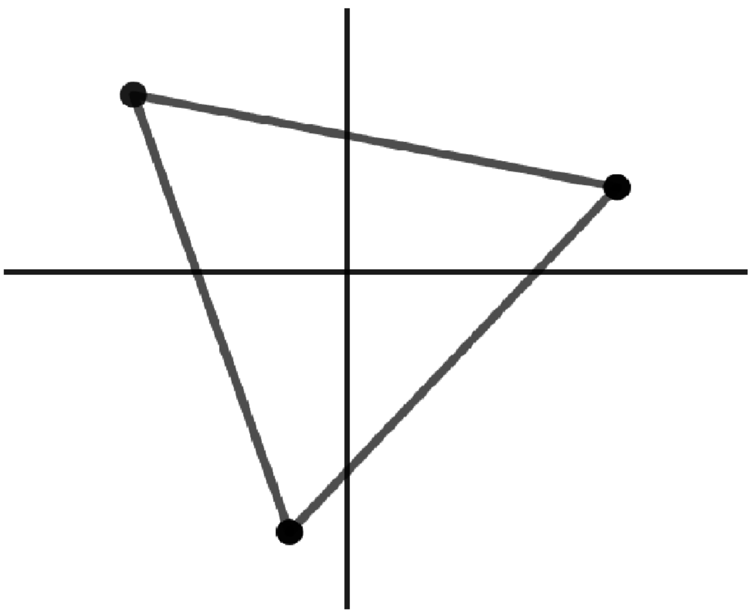
Way 5

M1: A correct strategy for the area of *ABC* using the "shoelace" method.

dM1: Completes the problem by linking the area of *DEF* correctly with *ABC*

A1: Correct value

Note the marks in (b) can be scored using inexact answers from (a) and the A1 scored if an exact area is obtained.



Question	Scheme	Marks	AOs
7(a)	$\lvert\mathbf{M}\rvert = 2(-k-8)+1(-3-12)+1(6-3k)=0 \Rightarrow k=\ldots$	M1	1.1b
	$k \neq -5$	A1	2.4
		(2)	
(b) Way 1	$\mathbf{M}=\begin{pmatrix}2 & -1 & 1\\ 3 & -6 & 4\\ 3 & 2 & -1\end{pmatrix} \Rightarrow \begin{pmatrix}x\\ y\\ z\end{pmatrix}=\mathbf{M}^{-1}\begin{pmatrix}p\\ 1\\ 0\end{pmatrix}$	M1	3.1a
	$\mathbf{M}^{-1}=\frac{1}{5}\begin{pmatrix}-2 & 1 & 2\\ 15 & -5 & -5\\ 24 & -7 & -9\end{pmatrix}$	B1	1.1b
	$\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\frac{1}{5}\begin{pmatrix}-2 & 1 & 2\\ 15 & -5 & -5\\ 24 & -7 & -9\end{pmatrix}\begin{pmatrix}p\\ 1\\ 0\end{pmatrix} \Rightarrow \begin{pmatrix}x\\ y\\ z\end{pmatrix}=\ldots$	M1	2.1
	$\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\frac{1}{5}\begin{pmatrix}-2p+1\\ 15p-5\\ 24p-7\end{pmatrix}$	A1	1.1b
	$\left(\frac{-2p+1}{5},\ 3p-1, \frac{24p-7}{5}\right)$	A1ft	2.5
		(5)	
(b) Way 2	$\begin{matrix}2x-\ y+\ z=p\\ 3x-6y+4z=1\\ 3x+2y-\ z=0\end{matrix} \Rightarrow \text{e.g.} \begin{matrix}8y-5z=-1\\ 9y-5z=3p-2\end{matrix} \Rightarrow y=\ldots$ $\Rightarrow x=\ldots, z=\ldots$	M1	3.1a
	$y=3p-1$(or $x=\frac{-2p+1}{5}$ or $z=\frac{24p-7}{5}$)	B1	1.1b
	$8(3p-1)-5z=-1 \Rightarrow z=\ldots \Rightarrow x=\ldots$	M1	2.1
	$z=\frac{24p-7}{5}$, $x=\frac{-2p+1}{5}$	A1	1.1b
	$\left(\frac{-2p+1}{5},\ 3p-1, \frac{24p-7}{5}\right)$	A1ft	2.5

(c)(i)	For consistency: E.g. $5x+y=4-q$ and $15x+3y=q$	M1	3.1a
	$4-q=\frac{q}{3}\Rightarrow q=...$	M1	2.1
	$q=3$	A1	1.1b
	Alternative for (c)(i): $x=1\Rightarrow 2-y+z=1,\ 3+2y-z=0\Rightarrow y=...,z=...$ M1 for allocating a number to one variable and solves for the other 2 $x=1, y=-4, z=-5\Rightarrow 3+20-20=q$ M1 substitutes into the second equation and solves for *q* A1: *q* = 3		
(ii)	Three **planes** that intersect in a **line** Or Three **planes** that form a **sheaf** allow **sheath**!	B1	2.4
		(4)	
			(11 marks)

Notes

(a)

M1: Attempts determinant, equates to zero and attempts to solve for *k* in order to establish the restriction for *k*. For the determinant, at least 2 of the 3 “elements” should be correct.

May see rule of Sarrus used for determinant e.g.

$$|\mathbf{M}|=(2)(k)(-1)+(4)(3)(-1)+(3)(2)(1)-(3)(k)(-1)-(2)(4)(2)-(-1)(3)(-1)=0\Rightarrow k=...$$

A1: Describes the correct condition for *k* with no contradictions. Allow e.g. $k < -5$, $k > -5$

(b)**Way 1**

M1: A complete strategy for solving the given equations. Need to see an attempt at the inverse followed by a correct method for finding *x*, *y* and *z*

B1: Correct inverse matrix

M1: Uses their inverse and attempts the multiplication with the correct vector

A1: Correct values for *x*, *y* and *z* in any form

A1ft: Correct values given in coordinate form only. **Follow through their *x*, *y* and *z*.**

Way 2

M1: A complete strategy for solving the given equations. Need to see an attempt at eliminating one variable followed by a correct method for finding *x*, *y* and *z*

B1: One correct value

M1: Uses the equations to find values for the other 2 variables

A1: Correct values for *x*, *y* and *z* in any form

A1ft: Correct values given in coordinate form only. **Follow through their *x*, *y* and *z*.**

(c)(i)

M1: Uses a correct strategy that will lead to establishing a value for *q*. E.g. eliminating one of *x*, *y* or *z*

M1: Solves a suitable equation to obtain a value for *q*

A1: Correct value

(ii)

B1: Describes the correct geometrical configuration.

Must include the **two** ideas of **planes** and meeting in a **line** or forming a **sheaf** with no contradictory statements.

Question	Scheme	Marks	AOs
8(a)	$k = 2.6$	B1	3.4
		(1)	
(b)	$x = 1.18 \Rightarrow \ln\left(3.6\times1.18 - "2.6"\right) = ...$	M1	1.1b
	$h = 0.4995...$ m	A1	2.2b
		(2)	
(c)	$y = \ln\left(3.6x - 2.6\right) \Rightarrow x = \dfrac{e^y + 2.6}{3.6}$ or $\dfrac{5e^y + 13}{18}$	B1ft	1.1a
	$V = \pi\int \left(\dfrac{e^y + 2.6}{3.6}\right)^2 dy = \dfrac{\pi}{3.6^2}\int \left(e^{2y} + 5.2e^y + 6.76\right)dy$ or $\dfrac{\pi}{324}\int \left(25e^{2y} + 130e^y + 169\right)dy$	M1	3.3
	$= \dfrac{\pi}{3.6^2}\left[\dfrac{1}{2}e^{2y} + 5.2e^y + 6.76y\right]$ $\left(\text{or } \dfrac{\pi}{324}\left[\dfrac{25}{2}e^{2y} + 130e^y + 169y\right]\right)$	A1	1.1b
	$= \dfrac{\pi}{3.6^2}\left\{\left(\dfrac{1}{2}e^{2h} + 5.2e^h + 6.76h\right) - \left(\dfrac{1}{2}e^0 + 5.2e^0 + 6.76(0)\right)\right\}$ or e.g. $= \dfrac{\pi}{324}\left\{\left(\dfrac{25}{2}e^{2h} + 130e^h + 169h\right) - \left(\dfrac{25}{2}e^0 + 130e^0 + 6.76(0)\right)\right\}$	M1	2.1
	$= \dfrac{\pi}{3.6^2}\left(\dfrac{1}{2}e^{2h} + 5.2e^h + 6.76h - 5.7\right)$	A1	1.1b
		(5)	
(d)	$\dfrac{dV}{dh} = \dfrac{\pi}{3.6^2}\left(e^{2h} + 5.2e^h + 6.76\right) = \dfrac{\pi}{3.6^2}\left(e^{0.4} + 5.2e^{0.2} + 6.76\right)$	M1	3.1a
	$\dfrac{dh}{dt} = \dfrac{dh}{dV}\dfrac{dV}{dt} = \dfrac{1}{3.539...}\times0.015\times60$	M1	1.1b
	$\dfrac{dh}{dt} = 25.4\,\text{cm}\,\text{h}^{-1}$	A1	3.2a
		(3)	
(d) Way 2	$y = 0.2 \Rightarrow x = \dfrac{2.6 + e^{0.2}}{3.6} \Rightarrow A = \pi\left(\dfrac{2.6 + e^{0.2}}{3.6}\right)^2 (= 3.54)$	M1	3.1a
	$\dfrac{dh}{dt} = \dfrac{0.015\times60}{3.54}$	M1	1.1b
	$\dfrac{dh}{dt} = 25.4\,\text{cm}\,\text{h}^{-1}$	A1	3.2a
			(11 marks)

Notes

(a)
B1: Uses the model to obtain a correct value for *k*. Must be 2.6 not -2.6
(b)

M1: Substitutes their value of k and $x = 1.18$ into the given model to find a value for y
A1: Infers that the depth of the pool could be awrt 0.5 m
(c)
B1ft: Uses the model to obtain x correctly in terms of y (follow through their k)
M1: Uses the model to obtain an expression for the volume of the pool using $\pi\int\left(their\ f(y)\right)^2\,\mathrm{d}y$ – must expand in order to reach an integrable form (allow poor squaring e.g. $(a + b)^2 = a^2 + b^2$. **Note that the π may be recovered later.**
A1: Correct integration
M1: Selects limits appropriate to the model (h and 0) substitutes and clearly shows the use of both limits (i.e. including zero)
A1: Correct expression **(allow unsimplified and isw if necessary)**
(d)
Way 1
M1: Recognises that $\frac{\mathrm{d}V}{\mathrm{d}h}$ is required and attempts to find $\frac{\mathrm{d}V}{\mathrm{d}h}$ or $\frac{\mathrm{d}h}{\mathrm{d}V}$ from their integration or using the earlier result (before integrating). Must clearly be identified as $\frac{\mathrm{d}V}{\mathrm{d}h}$ or $\frac{\mathrm{d}h}{\mathrm{d}V}$ unless this implied by subsequent work.
M1: Evidence of the correct use of the chain rule (ignore any confusion with units). Look for an attempt to divide 15 or their converted 15 by their $\frac{\mathrm{d}V}{\mathrm{d}h}$ or to multiply 15 or their converted 15 by $\frac{\mathrm{d}h}{\mathrm{d}V}$ **but must reach a <u>value</u> for $\frac{\mathrm{d}h}{\mathrm{d}t}$ <u>but you do not need to check their value.</u>**
A1: Interprets their solution correctly to obtain the correct answer (awrt 25.4) **with the correct units**
Way 2
M1: Uses $y = 0.2$ to find x and the surface area of the water at that instant
M1: Attempts to divide the rate by their area (ignore any confusion with units)
A1: Interprets their solution correctly to obtain the correct answer (awrt 25.4) **with the correct units**

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