14.

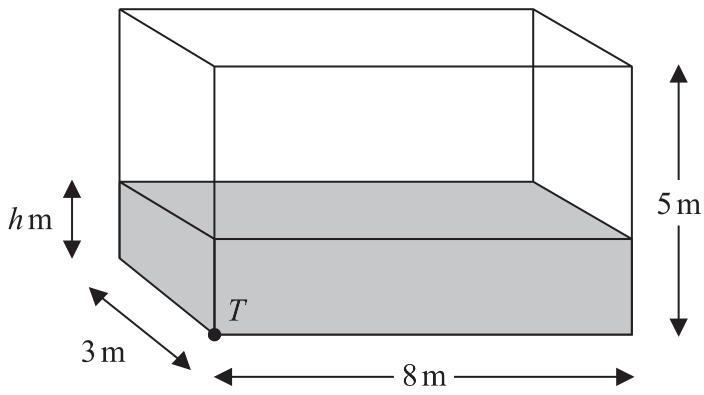

Figure 5

Water flows at a constant rate into a large tank.

The tank is a cuboid, with all sides of negligible thickness.

The base of the tank measures 8 m by 3 m and the height of the tank is 5 m.

There is a tap at a point T at the bottom of the tank, as shown in Figure 5.

At time t minutes after the tap has been opened

- the depth of water in the tank is h metres
- water is flowing into the tank at a constant rate of 0.48 m^3 per minute
- water is modelled as leaving the tank through the tap at a rate of $0.1h$ m^3 per minute

(a) Show that, according to the model,

$$1200\frac{dh}{dt} = 24 - 5h$$

(4)

Given that when the tap was opened, the depth of water in the tank was 2 m,

(b) show that, according to the model,

$$h = A + Be^{-kt}$$

where A, B and k are constants to be found.

(6)

Given that the tap remains open,

(c) determine, according to the model, whether the tank will ever become full, giving a reason for your answer.

(2)

10. The height above ground, H metres, of a passenger on a roller coaster can be modelled by the differential equation

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{H\cos(0.25t)}{40}$$

where t is the time, in seconds, from the start of the ride.

Given that the passenger is 5 m above the ground at the start of the ride,

(a) show that $H = 5\mathrm{e}^{0.1\sin(0.25t)}$

(5)

(b) State the maximum height of the passenger above the ground.

(1)

The passenger reaches the maximum height, for the second time, T seconds after the start of the ride.

(c) Find the value of T.

(2)

10. A spherical mint of radius 5 mm is placed in the mouth and sucked.
Four minutes later, the radius of the mint is 3 mm.

In a simple model, the rate of decrease of the radius of the mint is inversely proportional to the square of the radius.

Using this model and all the information given,

(a) find an equation linking the radius of the mint and the time.
(You should define the variables that you use.)

(5)

(b) Hence find the total time taken for the mint to completely dissolve. Give your answer in minutes and seconds to the nearest second.

(2)

(c) Suggest a limitation of the model.

(1)

14. A scientist is studying a population of mice on an island.

The number of mice, N, in the population, t months after the start of the study, is modelled by the equation

$$N = \frac{900}{3 + 7e^{-0.25t}}, \quad t \in \mathbb{R}, \quad t \geqslant 0$$

(a) Find the number of mice in the population at the start of the study.

(1)

(b) Show that the rate of growth $\frac{dN}{dt}$ is given by $\frac{dN}{dt} = \frac{N(300 - N)}{1200}$

(4)

The rate of growth is a maximum after T months.

(c) Find, according to the model, the value of T.

(4)

According to the model, the maximum number of mice on the island is P.

(d) State the value of P.

(1)

14. (a) Use the substitution $u = 4 - \sqrt{h}$ to show that

$$\int \frac{\mathrm{d}h}{4-\sqrt{h}} = -8\ln\left|4-\sqrt{h}\right| - 2\sqrt{h} + k$$

where k is a constant

(6)

A team of scientists is studying a species of slow growing tree.

The rate of change in height of a tree in this species is modelled by the differential equation

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{t^{0.25}\left(4-\sqrt{h}\right)}{20}$$

where h is the height in metres and t is the time, measured in years, after the tree is planted.

(b) Find, according to the model, the range in heights of trees in this species.

(2)

One of these trees is one metre high when it is first planted.

According to the model,

(c) calculate the time this tree would take to reach a height of 12 metres, giving your answer to 3 significant figures.

(7)

8. A new smartphone was released by a company.

The company monitored the total number of phones sold, n, at time t days after the phone was released.

The company observed that, during this time,

the rate of increase of n was proportional to n

Use this information to write down a suitable equation for n in terms of t.

(*You do not need to evaluate any unknown constants in your equation*.)

(2)

14. A large spherical balloon is deflating.

At time t seconds the balloon has radius r cm and volume V cm^3

The volume of the balloon is modelled as decreasing at a constant rate.

(a) Using this model, show that

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -\frac{k}{r^2}$$

where k is a positive constant.

(3)

Given that

- the initial radius of the balloon is 40 cm
- after 5 seconds the radius of the balloon is 20 cm
- the volume of the balloon continues to decrease at a constant rate until the balloon is empty

(b) solve the differential equation to find a complete equation linking r and t.

(5)

(c) Find the limitation on the values of t for which the equation in part (b) is valid.

(2)

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9. (a) Use the substitution $u = 4 - \sqrt{x}$ to find

$$\int \frac{dx}{4 - \sqrt{x}}$$

(6)

A team of scientists is studying a species of slow growing tree.

The rate of change in height of a tree in this species is modelled by the differential equation

$$\frac{dh}{dt} = \frac{4 - \sqrt{h}}{20}$$

where h is the height in metres and t is the time measured in years after the tree is planted.

(b) Find the range in values of h for which the height of a tree in this species is increasing.
(2)

(c) Given that one of these trees is 1 metre high when it is planted, calculate the time it would take to reach a height of 10 metres. Write your answer to 3 significant figures.
(7)

Leave blank

6. (a) Express $\dfrac{5-4x}{(2x-1)(x+1)}$ in partial fractions.

(3)

(b) (i) Find a general solution of the differential equation

$$(2x-1)(x+1)\frac{dy}{dx} = (5-4x)y, \quad x > \frac{1}{2}$$

Given that $y = 4$ when $x = 2$,

(ii) find the particular solution of this differential equation. Give your answer in the form $y = f(x)$.

(7)

Leave blank

8. (a) Prove by differentiation that

$$\frac{\mathrm{d}}{\mathrm{d}y}(\ln \tan 2y) = \frac{4}{\sin 4y}, \qquad 0 < y < \frac{\pi}{4}$$

(4)

(b) Given that $y = \frac{\pi}{6}$ when $x = 0$, solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2\cos x \sin 4y, \qquad 0 < y < \frac{\pi}{4}$$

Give your answer in the form $\tan 2y = A\mathrm{e}^{B\sin x}$, where A and B are constants to be determined.

(6)

Leave blank

9. (a) Express $\dfrac{3x^2 - 4}{x^2(3x - 2)}$ in partial fractions.

(4)

(b) Given that $x > \dfrac{2}{3}$, find the general solution of the differential equation

$$x^2(3x - 2)\,\frac{dy}{dx} = y(3x^2 - 4)$$

Give your answer in the form $y = f(x)$.

(6)

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13.

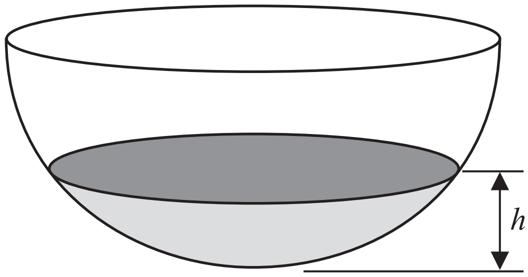

Figure 4

Figure 4 shows a hemispherical bowl containing some water.

At t seconds, the height of the water is h cm and the volume of the water is V cm^3, where

$$V = \frac{1}{3}\pi h^2(30 - h), \qquad 0 < h \leqslant 10$$

The water is leaking from a hole in the bottom of the bowl.

Given that $\dfrac{dV}{dt} = -\dfrac{1}{10}V$

(a) show that $\dfrac{dh}{dt} = -\dfrac{h(30 - h)}{30(20 - h)}$ **(5)**

(b) Write $\dfrac{30(20 - h)}{h(30 - h)}$ in partial fraction form. **(3)**

Given that $h = 10$ when $t = 0$,

(c) use your answers to parts (a) and (b) to find the time taken for the height of the water to fall to 5 cm. Give your answer in seconds to 2 decimal places. **(6)**

Leave blank

12. In freezing temperatures, ice forms on the surface of the water in a barrel. At time t hours after the start of freezing, the thickness of the ice formed is x mm. You may assume that the thickness of the ice is uniform across the surface of the water.

At 4pm there is no ice on the surface, and freezing begins.

At 6pm, after two hours of freezing, the ice is 1.5 mm thick.

In a simple model, the rate of increase of x, in mm per hour, is assumed to be constant for a period of 20 hours.

Using this simple model,

(a) express t in terms of x,

(2)

(b) find the value of t when $x = 3$

(1)

In a second model, the rate of increase of x, in mm per hour, is given by the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\lambda}{(2x+1)} \quad \text{where } \lambda \text{ is a constant and } 0 \leqslant t \leqslant 20$$

Using this second model,

(c) solve the differential equation and express t in terms of x and λ,

(3)

(d) find the exact value for λ,

(1)

(e) find at what time the ice is predicted to be 3 mm thick.

(2)

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12.

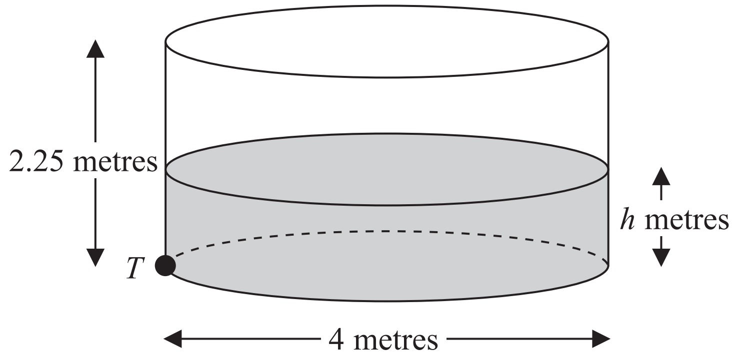

Figure 4

Figure 4 shows a right cylindrical water tank. The diameter of the circular cross section of the tank is 4 m and the height is 2.25 m. Water is flowing into the tank at a constant rate of 0.4π $m^3 min^{-1}$. There is a tap at a point T at the base of the tank. When the tap is open, water leaves the tank at a rate of $0.2\pi\sqrt{h}$ $m^3 min^{-1}$, where h is the height of the water in metres.

(a) Show that at time t minutes after the tap has been opened, the height h m of the water in the tank satisfies the differential equation

$$20\frac{dh}{dt} = 2 - \sqrt{h}$$

(5)

At the instant when the tap is opened, $t = 0$ and $h = 0.16$

(b) Use the differential equation to show that the time taken to fill the tank to a height of 2.25 m is given by

$$\int_{0.16}^{2.25} \frac{20}{2 - \sqrt{h}} \, dh$$

(2)

Using the substitution $h = (2 - x)^2$, or otherwise,

(c) find the time taken to fill the tank to a height of 2.25 m.

Give your answer in minutes to the nearest minute.

(*Solutions based entirely on graphical or numerical methods are not acceptable.*)

(7)

Leave blank

11. (a) Given $0 \leqslant h < 25$, use the substitution $u = 5 - \sqrt{h}$ to show that

$$\int \frac{\mathrm{d}h}{5-\sqrt{h}} = -10\ln\left(5-\sqrt{h}\right) - 2\sqrt{h} + k$$

where k is a constant.

(6)

A team of scientists is studying a species of tree.

The rate of change in height of a tree of this species is modelled by the differential equation

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{t^{0.2}\left(5-\sqrt{h}\right)}{5}$$

where h is the height of the tree in metres and t is the time in years after the tree is planted.

One of these trees is 2 metres high when it is planted.

(b) Use integration to calculate the time it would take for this tree to reach a height of 15 metres, giving your answer to one decimal place.

(7)

(c) Hence calculate the rate of change in height of this tree when its height is 15 metres. Write your answer in centimetres per year to the nearest centimetre.

(1)

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7.

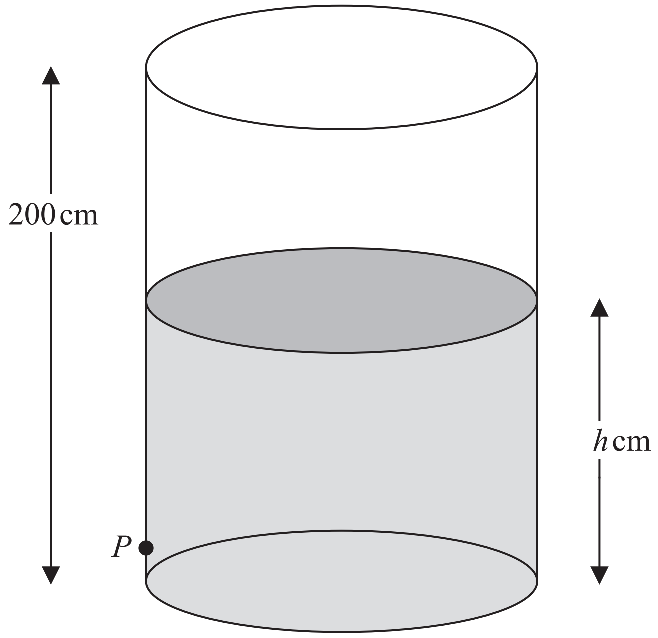

Figure 3

Figure 3 shows a vertical cylindrical tank of height 200 cm containing water.
Water is leaking from a hole P on the side of the tank.

At time t minutes after the leaking starts, the height of water in the tank is h cm.

The height h cm of the water in the tank satisfies the differential equation

$$\frac{dh}{dt} = k(h-9)^{\frac{1}{2}}, \qquad 9 < h \leqslant 200$$

where k is a constant.

Given that, when $h = 130$, the height of the water is falling at a rate of 1.1 cm per minute,

(a) find the value of k.

(2)

Given that the tank was full of water when the leaking started,

(b) solve the differential equation with your value of k, to find the value of t when $h = 50$

(6)

Leave blank

7. (a) Express $\dfrac{2}{P(P-2)}$ in partial fractions.

(3)

A team of biologists is studying a population of a particular species of animal.

The population is modelled by the differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{1}{2}P(P-2)\cos 2t, \quad t \geqslant 0$$

where P is the population in thousands, and t is the time measured in years since the start of the study.

Given that $P = 3$ when $t = 0$,

(b) solve this differential equation to show that

$$P = \frac{6}{3 - \mathrm{e}^{\frac{1}{2}\sin 2t}}$$

(7)

(c) find the time taken for the population to reach 4000 for the first time.
Give your answer in years to 3 significant figures.

(3)

Leave blank

8. In an experiment testing solid rocket fuel, some fuel is burned and the waste products are collected. Throughout the experiment the sum of the masses of the unburned fuel and waste products remains constant.

Let x be the mass of waste products, in kg, at time t minutes after the start of the experiment. It is known that at time t minutes, the rate of increase of the mass of waste products, in kg per minute, is k times the mass of unburned fuel remaining, where k is a positive constant.

The differential equation connecting x and t may be written in the form

$$\frac{dx}{dt} = k(M - x), \text{ where } M \text{ is a constant.}$$

(a) Explain, in the context of the problem, what $\frac{dx}{dt}$ and M represent.
(2)

Given that initially the mass of waste products is zero,

(b) solve the differential equation, expressing x in terms of k, M and t.
(6)

Given also that $x = \frac{1}{2}M$ when $t = \ln 4$,

(c) find the value of x when $t = \ln 9$, expressing x in terms of M, in its simplest form.
(4)