3. $f(z) = z^4 + az^3 + 6z^2 + bz + 65$

where a and b are real constants.

Given that $z = 3 + 2\mathbf{i}$ is a root of the equation $f(z) = 0$, show the roots of $f(z) = 0$ on a single Argand diagram.

(9)

Answer ALL questions. Write your answers in the spaces provided.

1. The roots of the equation

$$x^3 - 8x^2 + 28x - 32 = 0$$

are α, β and γ

Without solving the equation, find the value of

(i) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

(ii) $(\alpha + 2)(\beta + 2)(\gamma + 2)$

(iii) $\alpha^2 + \beta^2 + \gamma^2$

(8)

Answer ALL questions. Write your answers in the spaces provided.

1.

$$f(z) = z^4 + az^3 + bz^2 + cz + d$$

where a, b, c and d are real constants.

Given that $-1 + 2i$ and $3 - i$ are two roots of the equation $f(z) = 0$

(a) show all the roots of $f(z) = 0$ on a single Argand diagram,

(4)

(b) find the values of a, b, c and d.

(5)

2. The roots of the equation

$$x^3 - 2x^2 + 4x - 5 = 0$$

are p, q and r.

Without solving the equation, find the value of

(i) $\frac{2}{p} + \frac{2}{q} + \frac{2}{r}$

(ii) $(p - 4)(q - 4)(r - 4)$

(iii) $p^3 + q^3 + r^3$

(8)

1. $f(z) = 3z^3 + pz^2 + 57z + q$

where p and q are real constants.

Given that $3 - 2\sqrt{2}i$ is a root of the equation $f(z) = 0$

(a) show all the roots of $f(z) = 0$ on a single Argand diagram,

(7)

(b) find the value of p and the value of q.

(3)

3. The cubic equation

$$ax^3 + bx^2 - 19x - b = 0$$

where a and b are constants, has roots α, β and γ

The cubic equation

$$w^3 - 9w^2 - 97w + c = 0$$

where c is a constant, has roots $(4\alpha - 1)$, $(4\beta - 1)$ and $(4\gamma - 1)$

Without solving either cubic equation, determine the value of a, the value of b and the value of c.

(6)

6. The cubic equation

$$4x^3 + px^2 - 14x + q = 0$$

where p and q are real positive constants, has roots α, β and γ

Given that $\alpha^2 + \beta^2 + \gamma^2 = 16$

(a) show that $p = 12$

(3)

Given that $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{14}{3}$

(b) determine the value of q

(3)

Without solving the cubic equation,

(c) determine the value of $(\alpha - 1)(\beta - 1)(\gamma - 1)$

(4)

1. The cubic equation

$$x^3 - 7x^2 - 12x + 6 = 0$$

has roots α, β and γ.

Without solving the equation, determine a cubic equation whose roots are $(\alpha + 2)$, $(\beta + 2)$ and $(\gamma + 2)$, giving your answer in the form $w^3 + pw^2 + qw + r = 0$, where p, q and r are integers to be found.

(5)

8. Given that a cubic equation has three distinct roots that all lie on the same straight line in the complex plane,

(a) describe the possible lines the roots can lie on.

(2)

$$f(z) = 8z^3 + bz^2 + cz + d$$

where b, c and d are real constants.

The roots of $f(z)$ are distinct and lie on a straight line in the complex plane.

Given that one of the roots is $\frac{3}{2} + \frac{3}{2}i$

(b) state the other two roots of $f(z)$

(1)

$$g(z) = z^3 + Pz^2 + Qz + 12$$

where P and Q are real constants, has 3 distinct roots.

The roots of $g(z)$ lie on a different straight line in the complex plane than the roots of $f(z)$

Given that

- $f(z)$ and $g(z)$ have one root in common
- one of the roots of $g(z)$ is -4

(c) (i) write down the value of the common root,

(1)

(ii) determine the value of the other root of $g(z)$

(3)

(d) Hence solve the equation $f(z) = g(z)$

(4)

2. The cubic equation

$$z^3 - 3z^2 + z + 5 = 0$$

has roots α, β and γ.

Without solving the equation, find the cubic equation whose roots are $(2\alpha + 1)$, $(2\beta + 1)$ and $(2\gamma + 1)$, giving your answer in the form $w^3 + pw^2 + qw + r = 0$, where p, q and r are integers to be found.

(5)

2. The cubic equation

$$2x^3 + 6x^2 - 3x + 12 = 0$$

has roots α, β and γ.

Without solving the equation, find the cubic equation whose roots are $(\alpha + 3)$, $(\beta + 3)$ and $(\gamma + 3)$, giving your answer in the form $pw^3 + qw^2 + rw + s = 0$, where p, q, r and s are integers to be found.

(5)

7. $$f(z) = z^3 - 8z^2 + pz - 24$$

where p is a real constant.

Given that the equation $f(z) = 0$ has distinct roots

$$\alpha,\ \beta \text{ and } \left(\alpha + \frac{12}{\alpha} - \beta\right)$$

(a) solve completely the equation $f(z) = 0$

(6)

(b) Hence find the value of p.

(2)

7.

$$f(z) = z^4 + az^3 + bz^2 + cz + d$$

where a, b, c and d are real constants.

The equation $f(z) = 0$ has complex roots z_1, z_2, z_3 and z_4
When plotted on an Argand diagram, the points representing z_1, z_2, z_3 and z_4 form the vertices of a square, with one vertex in each quadrant.
Given that $z_1 = 2 + 3i$, determine the values of a, b, c and d.

(6)

9. The cubic equation

$$3x^3 + x^2 - 4x + 1 = 0$$

has roots α, β, and γ.

Without solving the cubic equation,

(a) determine the value of $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

(3)

(b) find a cubic equation that has roots $\frac{1}{\alpha}$, $\frac{1}{\beta}$ and $\frac{1}{\gamma}$, giving your answer in the form

$x^3 + ax^2 + bx + c = 0$, where a, b and c are integers to be determined.

(3)

2. The cubic equation

$$9x^3 - 5x^2 + 4x + 7 = 0$$

has roots α, β and γ.

Without solving the equation, find the cubic equation whose roots are $(3\alpha - 2)$, $(3\beta - 2)$ and $(3\gamma - 2)$, giving your answer in the form $aw^3 + bw^2 + cw + d = 0$, where a, b, c and d are integers to be determined.

(5)

7. $f(z) = z^4 - 6z^3 + pz^2 + qz + r$

where p, q and r are real constants.

The roots of the equation $f(z) = 0$ are α, β, γ and δ where $\alpha = 3$ and $\beta = 2 + i$

Given that γ is a complex root of $f(z) = 0$

(a) (i) write down the root γ,

(ii) explain why δ must be real.

(2)

(b) Determine the value of δ.

(2)

(c) Hence determine the values of p, q and r.

(3)

(d) Write down the roots of the equation $f(-2z) = 0$

(2)

4. The roots of the quartic equation

$$3x^4 + 5x^3 - 7x + 6 = 0$$

are α, β, γ and δ

Making your method clear and without solving the equation, determine the exact value of

(i) $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$

(3)

(ii) $\dfrac{2}{\alpha} + \dfrac{2}{\beta} + \dfrac{2}{\gamma} + \dfrac{2}{\delta}$

(3)

(iii) $(3 - \alpha)(3 - \beta)(3 - \gamma)(3 - \delta)$

(3)

2. $f(z) = z^3 + az^2 + bz + 175$ where a and b are real constants

Given that $-3 + 4i$ is a root of the equation $f(z) = 0$

(a) determine the value of a and the value of b.

(4)

(b) Show all the roots of the equation $f(z) = 0$ on a single Argand diagram.

(2)

(c) Write down the roots of the equation $f(z + 2) = 0$

(1)

10. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

(i) The quartic equation

$$z^4 + 5z^2 - 30 = 0$$

has roots p, q, r and s.

Without solving the equation, determine the quartic equation whose roots are

$$(3p - 1),\ (3q - 1),\ (3r - 1) \text{ and } (3s - 1)$$

Give your answer in the form $w^4 + aw^3 + bw^2 + cw + d = 0$, where a, b, c and d are integers to be found.

(5)

(ii) The roots of the cubic equation

$$4x^3 + nx + 81 = 0 \qquad \text{where } n \text{ is a real constant}$$

are α, 2α and $\alpha - \beta$

Determine

(a) the values of the roots of the equation,

(5)

(b) the value of n.

(2)