3. (i)

$$\mathbf{M} = \begin{pmatrix} 2 & a & 4 \\ 1 & -1 & -1 \\ -1 & 2 & -1 \end{pmatrix}$$

where a is a constant.

(a) For which values of a does the matrix $\mathbf{M}$ have an inverse?

(2)

Given that $\mathbf{M}$ is non-singular,

(b) find $\mathbf{M}^{-1}$ in terms of a

(4)

(ii) Prove by induction that for all positive integers n,

$$\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^n = \begin{pmatrix} 3^n & 0 \\ 3(3^n - 1) & 1 \end{pmatrix}$$

(6)

7.

$$\mathbf{M} = \begin{pmatrix} 2 & -1 & 1 \\ 3 & k & 4 \\ 3 & 2 & -1 \end{pmatrix} \quad \text{where } k \text{ is a constant}$$

(a) Find the values of k for which the matrix $\mathbf{M}$ has an inverse.

(2)

(b) Find, in terms of p, the coordinates of the point where the following planes intersect

$$2x - y + z = p$$
$$3x - 6y + 4z = 1$$
$$3x + 2y - z = 0$$

(5)

(c) (i) Find the value of q for which the set of simultaneous equations

$$2x - y + z = 1$$
$$3x - 5y + 4z = q$$
$$3x + 2y - z = 0$$

can be solved.

(ii) For this value of q, interpret the solution of the set of simultaneous equations geometrically.

(4)

6.

$$\mathbf{M} = \begin{pmatrix} k & 5 & 7 \\ 1 & 1 & 1 \\ 2 & 1 & -1 \end{pmatrix} \quad \text{where } k \text{ is a constant}$$

(a) Given that $k \neq 4$, find, in terms of k, the inverse of the matrix $\mathbf{M}$.

(4)

(b) Find, in terms of p, the coordinates of the point where the following planes intersect.

$$\begin{aligned} 2x + 5y + 7z &= 1 \\ x + y + z &= p \\ 2x + y - z &= 2 \end{aligned}$$

(3)

(c) (i) Find the value of q for which the following planes intersect in a straight line.

$$\begin{aligned} 4x + 5y + 7z &= 1 \\ x + y + z &= q \\ 2x + y - z &= 2 \end{aligned}$$

(ii) For this value of q, determine a vector equation for the line of intersection.

(7)

4. (i) **A** is a 2 by 2 matrix and **B** is a 2 by 3 matrix.

Giving a reason for your answer, explain whether it is possible to evaluate

(a) **AB**

(b) **A** + **B**

(2)

(ii) Given that

$$\begin{pmatrix} -5 & 3 & 1 \\ a & 0 & 0 \\ b & a & b \end{pmatrix}\begin{pmatrix} 0 & 5 & 0 \\ 2 & 12 & -1 \\ -1 & -11 & 3 \end{pmatrix} = \lambda\mathbf{I}$$

where a, b and λ are constants,

(a) determine

- the value of λ
- the value of a
- the value of b

(b) Hence deduce the inverse of the matrix $\begin{pmatrix} -5 & 3 & 1 \\ a & 0 & 0 \\ b & a & b \end{pmatrix}$

(3)

(iii) Given that

$$\mathbf{M} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & \sin\theta & \cos\theta \\ 0 & \cos 2\theta & \sin 2\theta \end{pmatrix} \qquad \text{where } 0 \leqslant \theta < \pi$$

determine the values of θ for which the matrix **M** is singular.

(4)

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

5.

$$\mathbf{M} = \begin{pmatrix} a & 2 & -3 \\ 2 & 3 & 0 \\ 4 & a & 2 \end{pmatrix} \quad \text{where } a \text{ is a constant}$$

(a) Show that **M** is non-singular for all values of a.

(2)

(b) Determine, in terms of a, $\mathbf{M}^{-1}$

(4)

2. **In this question you must show all stages of your working.**

A college offers only three courses: Construction, Design and Hospitality.

Each student enrols on just one of these courses.

In 2019, there was a total of 1110 students at this college.

There were 370 more students enrolled on Construction than Hospitality.

In 2020 the number of students enrolled on

- Construction **increased** by 1.25%
- Design **increased** by 2.5%
- Hospitality **decreased** by 2%

In 2020, the total number of students at the college increased by 0.27% to 2 significant figures.

(a) (i) Define, for each course, a variable for the number of students enrolled on that course in 2019.

(ii) Using your variables from part (a)(i), write down **three** equations that model this situation.

(4)

(b) By forming and solving a matrix equation, determine how many students were enrolled on each of the three courses in 2019.

(4)

4. Prove by induction that for $n \in \mathbb{N}$

$$\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & -2n \\ 0 & 1 \end{pmatrix}$$

(5)

8. A colony of small mammals is being studied.
In the study, the mammals are divided into 3 categories

N (newborns)	0 to less than 1 month old
J (juveniles)	1 to 3 months old
B (breeders)	over 3 months old

(a) State one limitation of the model regarding the division into these categories.

(1)

A model for the population of the colony is given by the matrix equation

$$\begin{pmatrix} N_{n+1} \\ J_{n+1} \\ B_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 \\ a & b & 0 \\ 0 & 0.48 & 0.96 \end{pmatrix} \begin{pmatrix} N_n \\ J_n \\ B_n \end{pmatrix}$$

where a and b are constants, and N_n, J_n and B_n are the respective numbers of the mammals in each category n months after the start of the study.

At the start of the study the colony has breeders only, with no newborns or juveniles.

According to the model, after 2 months the number of newborns is 48 and the number of juveniles is 40

(b) (i) Determine the number of mammals in the colony at the start of the study.

(ii) Show that $a = 0.8$

(4)

(c) Determine, in terms of b,

$$\begin{pmatrix} 0 & 0 & 2 \\ 0.8 & b & 0 \\ 0 & 0.48 & 0.96 \end{pmatrix}^{-1}$$

(3)

Given that the model predicts approximately 1015 mammals **in total** at the start of a particular month, and approximately 596 **newborns**, 464 **juveniles** and 437 **breeders** at the start of the next month,

(d) determine the value of b, giving your answer to 2 decimal places.

(3)

It is decided to monitor the number of **newborn** males and females as a part of the study. Assuming that 42% of newborns are male,

(e) refine the matrix equation for the model to reflect this information, giving a reason for your answer.
(*There is no need to estimate any unknown values for the refined model, but any known values should be made clear.*)

(2)

Answer ALL questions. Write your answers in the spaces provided.

1.

$$\mathbf{M} = \begin{pmatrix} 2 & 1 & -3 \\ 4 & -2 & 1 \\ 3 & 5 & -2 \end{pmatrix}$$

(a) Find $\mathbf{M}^{-1}$ giving each element in exact form.

(2)

(b) Solve the simultaneous equations

$$2x + y - 3z = -4$$
$$4x - 2y + z = 9$$
$$3x + 5y - 2z = 5$$

(2)

(c) Interpret the answer to part (b) geometrically.

(1)

10. The population of chimpanzees in a particular country consists of juveniles and adults. Juvenile chimpanzees do not reproduce.

In a study, the numbers of juvenile and adult chimpanzees were estimated at the start of each year. A model for the population satisfies the matrix system

$$\begin{pmatrix} J_{n+1} \\ A_{n+1} \end{pmatrix} = \begin{pmatrix} a & 0.15 \\ 0.08 & 0.82 \end{pmatrix} \begin{pmatrix} J_n \\ A_n \end{pmatrix} \qquad n = 0,1,2,\ldots$$

where a is a constant, and J_n and A_n are the respective numbers of juvenile and adult chimpanzees n years after the start of the study.

(a) Interpret the meaning of the constant a in the context of the model.

(1)

At the start of the study, the total number of chimpanzees in the country was estimated to be 64 000

According to the model, after one year the number of juvenile chimpanzees is 15 360 and the number of adult chimpanzees is 43 008

(b) (i) Find, in terms of a

$$\begin{pmatrix} a & 0.15 \\ 0.08 & 0.82 \end{pmatrix}^{-1}$$

(3)

(ii) Hence, or otherwise, find the value of a.

(3)

(iii) Calculate the change in the number of juvenile chimpanzees in the first year of the study, according to this model.

(2)

Given that the number of juvenile chimpanzees is known to be in decline in the country,

(c) comment on the short-term suitability of this model.

(1)

A study of the population revealed that adult chimpanzees stop reproducing at the age of 40 years.

(d) Refine the matrix system for the model to reflect this information, giving a reason for your answer.

(*There is no need to estimate any unknown values for the refined model, but any known values should be made clear.*)

(2)

1. A system of three equations is defined by

$$\begin{aligned} kx + 3y - z &= 3 \\ 3x - y + z &= -k \\ -16x - ky - kz &= k \end{aligned}$$

where k is a positive constant.

Given that there is no unique solution to all three equations,

(a) show that $k = 2$

(2)

Using $k = 2$

(b) determine whether the three equations are consistent, justifying your answer.

(3)

(c) Interpret the answer to part (b) geometrically.

(1)

4. $\mathbf{M} = \begin{pmatrix} 2 & 1 & 4 \\ k & 2 & -2 \\ 4 & 1 & -2 \end{pmatrix}$ $\mathbf{N} = \begin{pmatrix} k-7 & 6 & -10 \\ 2 & -20 & 24 \\ -3 & 2 & -1 \end{pmatrix}$

where k is a constant.

(a) Determine, in simplest form in terms of k, the matrix $\mathbf{MN}$.

(2)

(b) Given that $k = 5$

(i) write down $\mathbf{MN}$

(ii) hence write down $\mathbf{M}^{-1}$

(2)

(c) Solve the simultaneous equations

$$2x + y + 4z = 2$$
$$5x + 2y - 2z = 3$$
$$4x + y - 2z = -1$$

(2)

(d) Interpret the answer to part (c) geometrically.

(1)

1.

$$\mathbf{A} = \begin{pmatrix} 4 & -1 \\ 7 & 2 \\ -5 & 8 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 2 & 3 & 2 \\ -1 & 6 & 5 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} -5 & 2 & 1 \\ 4 & 3 & 8 \\ -6 & 11 & 2 \end{pmatrix}$$

Given that **I** is the 3×3 identity matrix,

(a) (i) show that there is an integer k for which

$$\mathbf{AB} - 3\mathbf{C} + k\mathbf{I} = \mathbf{0}$$

stating the value of k

(ii) explain why there can be no constant m such that

$$\mathbf{BA} - 3\mathbf{C} + m\mathbf{I} = \mathbf{0}$$

(4)

(b) (i) Show how the matrix **C** can be used to solve the simultaneous equations

$$\begin{aligned} -5x + 2y + z &= -14 \\ 4x + 3y + 8z &= 3 \\ -6x + 11y + 2z &= 7 \end{aligned}$$

(ii) Hence use your calculator to solve these equations.

(3)

1.

$$\begin{pmatrix} x & 9 \\ y & z \end{pmatrix} - 3\begin{pmatrix} z & y \\ z & y \end{pmatrix} = k\mathbf{I}$$

where x, y, z and k are constants.

Determine the value of x, the value of y and the value of z.

(4)