7. Given that $k \in \mathbb{Z}^+$

(a) show that $\displaystyle\int_k^{3k} \frac{2}{(3x-k)}\,\mathrm{d}x$ is independent of k,

(4)

(b) show that $\displaystyle\int_k^{2k} \frac{2}{(2x-k)^2}\,\mathrm{d}x$ is inversely proportional to k.

(3)

13. Show that

$$\int_0^2 2x\sqrt{x+2}\,\mathrm{d}x = \frac{32}{15}\left(2+\sqrt{2}\right)$$

(7)

13.

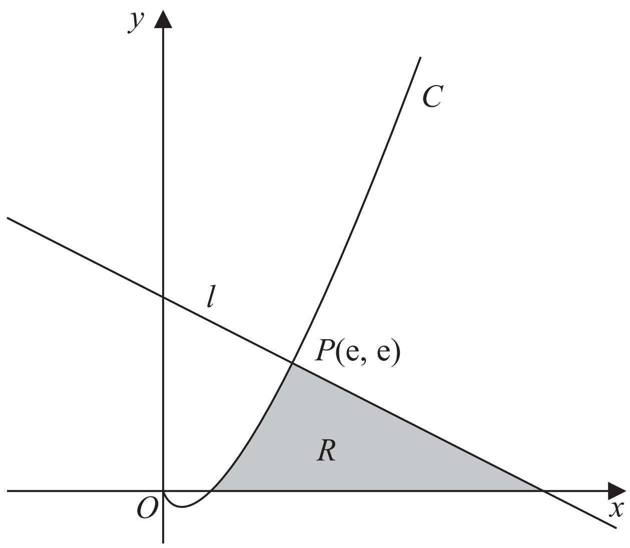

Figure 2

Figure 2 shows a sketch of part of the curve C with equation $y = x\ln x, \quad x > 0$

The line l is the normal to C at the point $P(\mathrm{e}, \mathrm{e})$

The region R, shown shaded in Figure 2, is bounded by the curve C, the line l and the x-axis.

Show that the exact area of R is $A\mathrm{e}^2 + B$ where A and B are rational numbers to be found.

(10)

8.

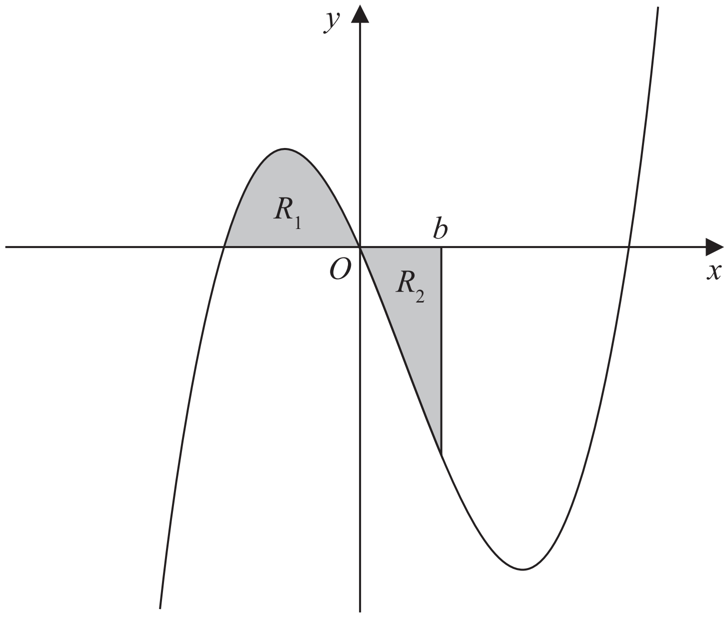

Figure 2

Figure 2 shows a sketch of part of the curve with equation $y = x(x + 2)(x - 4)$.

The region R_1 shown shaded in Figure 2 is bounded by the curve and the negative x-axis.

(a) Show that the exact area of R_1 is $\frac{20}{3}$

(4)

The region R_2 also shown shaded in Figure 2 is bounded by the curve, the positive x-axis and the line with equation $x = b$, where b is a positive constant and $0 < b < 4$

Given that the area of R_1 is equal to the area of R_2

(b) verify that b satisfies the equation

$$(b + 2)^2 (3b^2 - 20b + 20) = 0$$

(4)

The roots of the equation $3b^2 - 20b + 20 = 0$ are 1.225 and 5.442 to 3 decimal places. The value of b is therefore 1.225 to 3 decimal places.

(c) Explain, with the aid of a diagram, the significance of the root 5.442

(2)

13. The curve C with equation

$$y = \frac{p - 3x}{(2x - q)(x + 3)} \qquad x \in \mathbb{R}, x \neq -3, x \neq 2$$

where p and q are constants, passes through the point $\left(3, \frac{1}{2}\right)$ and has two vertical asymptotes with equations $x = 2$ and $x = -3$

(a) (i) Explain why you can deduce that $q = 4$

(ii) Show that $p = 15$

(3)

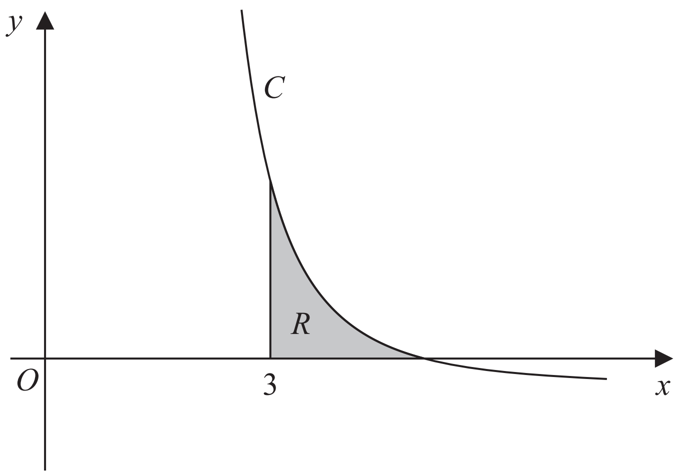

Figure 4

Figure 4 shows a sketch of part of the curve C. The region R, shown shaded in Figure 4, is bounded by the curve C, the x-axis and the line with equation $x = 3$

(b) Show that the exact value of the area of R is $a \ln 2 + b \ln 3$, where a and b are rational constants to be found.

(8)

DO NOT WRITE IN THIS AREA

5.

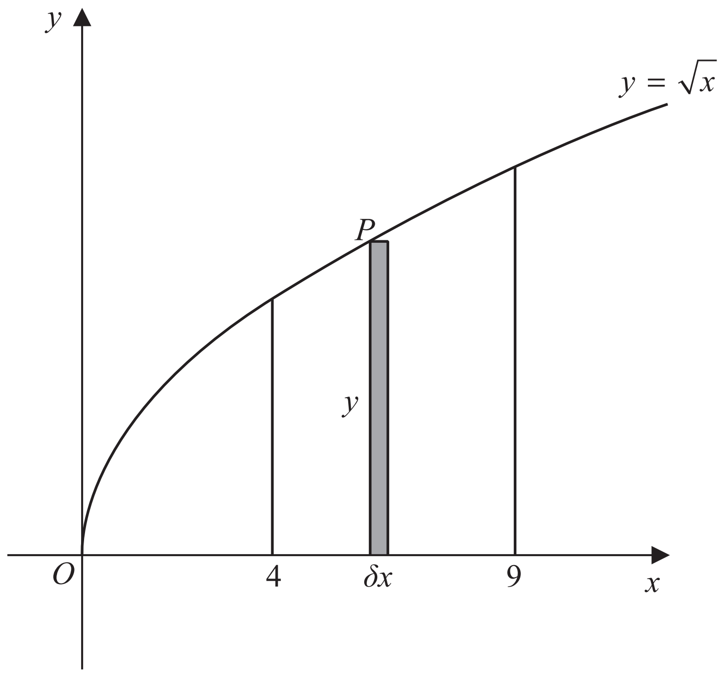

Figure 3

Figure 3 shows a sketch of the curve with equation $y = \sqrt{x}$

The point $P(x, y)$ lies on the curve.

The rectangle, shown shaded on Figure 3, has height y and width δx.

Calculate

$$\lim_{\delta x \to 0} \sum_{x=4}^{9} \sqrt{x}\, \delta x$$

(3)

10. (a) Use the substitution $x = u^2 + 1$ to show that

$$\int_5^{10} \frac{3\,\mathrm{d}x}{(x-1)\left(3+2\sqrt{x-1}\right)} = \int_p^q \frac{6\,\mathrm{d}u}{u(3+2u)}$$

where p and q are positive constants to be found.

(4)

(b) Hence, using algebraic integration, show that

$$\int_5^{10} \frac{3\,\mathrm{d}x}{(x-1)\left(3+2\sqrt{x-1}\right)} = \ln a$$

where a is a rational constant to be found.

(6)

6. (a) Given that

$$\frac{x^2 + 8x - 3}{x + 2} \equiv Ax + B + \frac{C}{x + 2} \qquad x \in \mathbb{R} \;\; x \neq -2$$

find the values of the constants A, B and C

(3)

(b) Hence, using algebraic integration, find the exact value of

$$\int_0^6 \frac{x^2 + 8x - 3}{x + 2}\,\mathrm{d}x$$

giving your answer in the form $a + b\ln 2$ where a and b are integers to be found.

(4)

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

8. A curve C has equation $y = \mathrm{f}(x)$

Given that

- $\mathrm{f}'(x) = 6x^2 + ax - 23$ where a is a constant
- the y intercept of C is -12
- $(x + 4)$ is a factor of $\mathrm{f}(x)$

find, in simplest form, $\mathrm{f}(x)$

(6)

Leave blank

3. Given that

$$4x^3 + 2x^2 + 17x + 8 \equiv (Ax + B)(x^2 + 4) + Cx + D$$

(a) find the values of the constants A, B, C and D.

(4)

(b) Hence find

$$\int_1^4 \frac{4x^3 + 2x^2 + 17x + 8}{x^2 + 4}\,dx$$

giving your answer in the form $p + \ln q$, where p and q are integers.

(6)

Leave blank

4. Find

(a) $\int (2x+3)^{12}\,dx$

(2)

(b) $\int \frac{5x}{4x^2+1}\,dx$

(2)

Leave blank

4. Use the substitution $x = 2\sin\theta$ to find the exact value of

$$\int_0^{\sqrt{3}} \frac{1}{(4-x^2)^{\frac{3}{2}}}\,dx$$

(7)

Leave blank

5. (i) Find the x coordinate of each point on the curve $y = \dfrac{x}{x+1}$, $x \neq -1$, at which the gradient is $\dfrac{1}{4}$

(4)

(ii) Given that

$$\int_{a}^{2a} \frac{t+1}{t}\,\mathrm{d}t = \ln 7 \qquad a > 0$$

find the exact value of the constant a.

(4)

Leave blank

8. $$f(\theta) = 9\cos^2\theta + \sin^2\theta$$

(a) Show that $f(\theta) = a + b\cos 2\theta$, where a and b are integers which should be found.

(3)

(b) Using your answer to part (a) and integration by parts, find the exact value of

$$\int_0^{\frac{\pi}{2}} \theta^2 f(\theta)\,d\theta$$

(6)

Leave blank

5. Use integration by parts to find the exact value of

$$\int_0^2 x2^x \, dx$$

Write your answer as a single simplified fraction.

(6)

Leave blank

2. Use integration by parts to find the exact value of $\int_1^{e} \frac{\ln x}{x^2}\,dx$

Write your answer in the form $a + \frac{b}{e}$, where a and b are integers.

(6)

DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA

Leave blank

5. (i) Find

$$\int \left((3x+5)^9 + e^{5x} \right) dx$$

(3)

(ii) Given that b is a constant greater than 2, and

$$\int_2^b \frac{x}{x^2+5}\,dx = \ln\left(\sqrt{6}\right)$$

use integration to find the value of b.

(5)

Leave blank

8. Use partial fractions, and integration, to find the exact value of $\displaystyle\int_3^4 \frac{2x^2 - 3}{x(x-1)}\,\mathrm{d}x$

Write your answer in the form $a + \ln b$, where a is an integer and b is a rational constant.
(8)

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

Leave blank

6. (i) Find

$$\int x\mathrm{e}^{4x}\,\mathrm{d}x$$

(3)

(ii) Find

$$\int \frac{8}{(2x-1)^3}\,\mathrm{d}x, \quad x > \frac{1}{2}$$

(2)

(iii) Given that $y = \frac{\pi}{6}$ at $x = 0$, solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^x \operatorname{cosec} 2y \operatorname{cosec} y$$

(7)

2. The speed of a small jet aircraft was measured every 5 seconds, starting from the time it turned onto a runway, until the time when it left the ground.

The results are given in the table below with the time in seconds and the speed in m s^{-1}.

Time (s)	0	5	10	15	20	25
Speed (m s^{-1})	2	5	10	18	28	42

Using all of this information,

(a) estimate the length of runway used by the jet to take off.

(3)

Given that the jet accelerated smoothly in these 25 seconds,

(b) explain whether your answer to part (a) is an underestimate or an overestimate of the length of runway used by the jet to take off.

(1)

DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA

1 The table below shows corresponding values of x and y for $y = \sqrt{\dfrac{x}{1+x}}$

The values of y are given to 4 significant figures.

x	0.5	1	1.5	2	2.5
y	0.5774	0.7071	0.7746	0.8165	0.8452

(a) Use the trapezium rule, with all the values of y in the table, to find an estimate for

$$\int_{0.5}^{2.5} \sqrt{\frac{x}{1+x}}\,\mathrm{d}x$$

giving your answer to 3 significant figures.

(3)

(b) Using your answer to part (a), deduce an estimate for $\displaystyle\int_{0.5}^{2.5} \sqrt{\frac{9x}{1+x}}\,\mathrm{d}x$

(1)

Given that

$$\int_{0.5}^{2.5} \sqrt{\frac{9x}{1+x}}\,\mathrm{d}x = 4.535 \text{ to 4 significant figures}$$

(c) comment on the accuracy of your answer to part (b).

(1)

Leave blank

9.

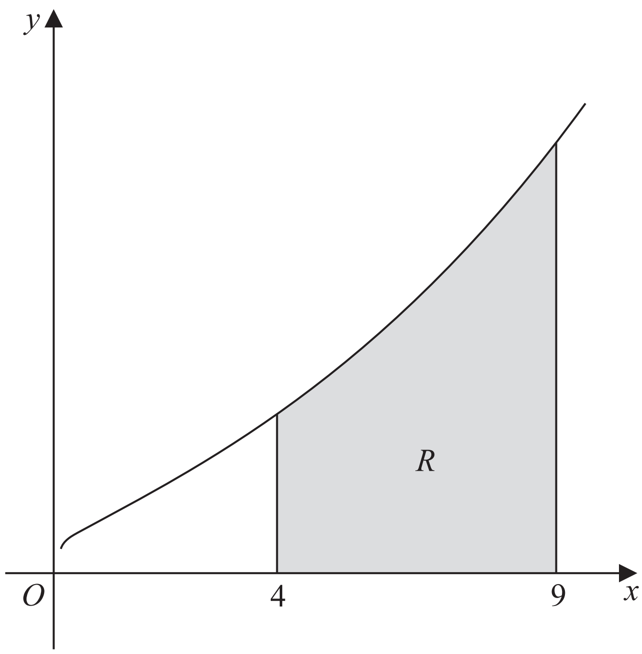

Figure 1

Figure 1 shows a sketch of part of the curve with equation $y = e^{\sqrt{x}}$, $x > 0$

The finite region R, shown shaded in Figure 1, is bounded by the curve, the x-axis and the lines $x = 4$ and $x = 9$

(a) Use the trapezium rule, with 5 strips of equal width, to obtain an estimate for the area of R, giving your answer to 2 decimal places.

(4)

(b) Use the substitution $u = \sqrt{x}$ to find, by integrating, the exact value for the area of R.

(7)

Leave blank

12.

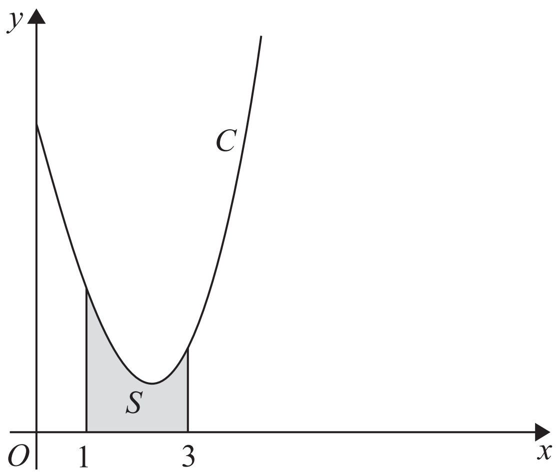

Figure 4

Figure 4 shows a sketch of part of the curve C with equation

$$y = \frac{x^2 \ln x}{3} - 2x + 4, \quad x > 0$$

The finite region S, shown shaded in Figure 4, is bounded by the curve C, the x-axis and the lines with equations $x = 1$ and $x = 3$

(a) Complete the table below with the value of y corresponding to $x = 2$. Give your answer to 4 decimal places.

x	1	1.5	2	2.5	3
y	2	1.3041		0.9089	1.2958

(1)

(b) Use the trapezium rule, with all the values of y in the completed table, to obtain an estimate for the area of S, giving your answer to 3 decimal places.

(3)

(c) Use calculus to find the exact area of S.

Give your answer in the form $\frac{a}{b} + \ln c$, where a, b and c are integers.

(6)

(d) Hence calculate the percentage error in using your answer to part (b) to estimate the area of S. Give your answer to one decimal place.

(2)

(e) Explain how the trapezium rule could be used to obtain a more accurate estimate for the area of S.

(1)

Leave blank

13.

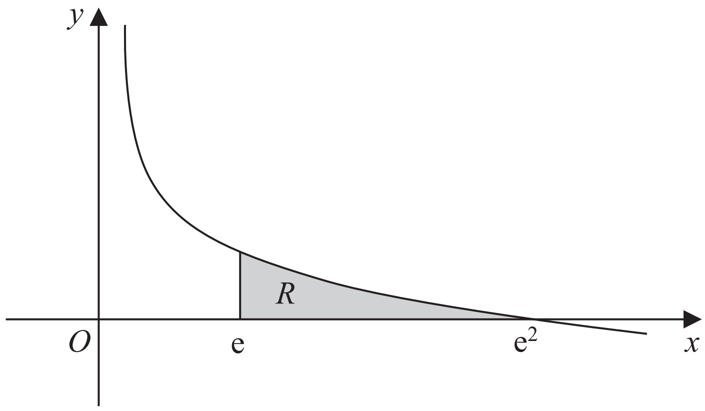

Figure 5

Figure 5 shows a sketch of part of the curve with equation $y = 2 - \ln x, \quad x > 0$

The finite region R, shown shaded in Figure 5, is bounded by the curve, the x-axis and the line with equation $x = \mathrm{e}$.

The table below shows corresponding values of x and y for $y = 2 - \ln x$

x	e	$\dfrac{\mathrm{e} + \mathrm{e}^2}{2}$	e^2
y	1		0

(a) Complete the table giving the value of y to 4 decimal places.

(1)

(b) Use the trapezium rule, with all the values of y in the completed table, to obtain an estimate for the area of R, giving your answer to 3 decimal places.

(3)

(c) Use integration by parts to show that $\int (\ln x)^2 \, \mathrm{d}x = x(\ln x)^2 - 2x \ln x + 2x + c$

(4)

The area R is rotated through 360° about the x-axis.

(d) Use calculus to find the exact volume of the solid generated.

Write your answer in the form $\pi \mathrm{e}(p\mathrm{e} + q)$, where p and q are integers to be found.

(6)

Leave blank

7.

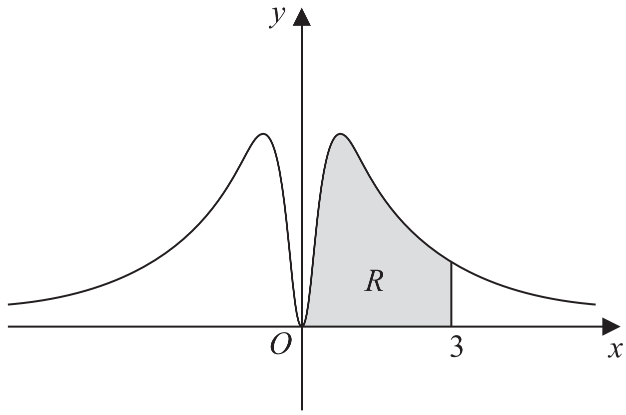

Figure 3

Figure 3 shows part of the curve C with equation

$$y = \frac{3\ln(x^2+1)}{(x^2+1)}, \quad x \in \mathbb{R}$$

(a) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$

(2)

(b) Using your answer to (a), find the exact coordinates of the stationary point on the curve C for which $x > 0$. Write each coordinate in its simplest form.

(5)

The finite region R, shown shaded in Figure 3, is bounded by the curve C, the x-axis and the line $x = 3$

(c) Complete the table below with the value of y corresponding to $x = 1$

x	0	1	2	3
y	0		$\frac{3}{5}\ln 5$	$\frac{3}{10}\ln 10$

(1)

(d) Use the trapezium rule with all the y values in the completed table to find an approximate value for the area of R, giving your answer to 4 significant figures.

(3)

Leave blank

7.

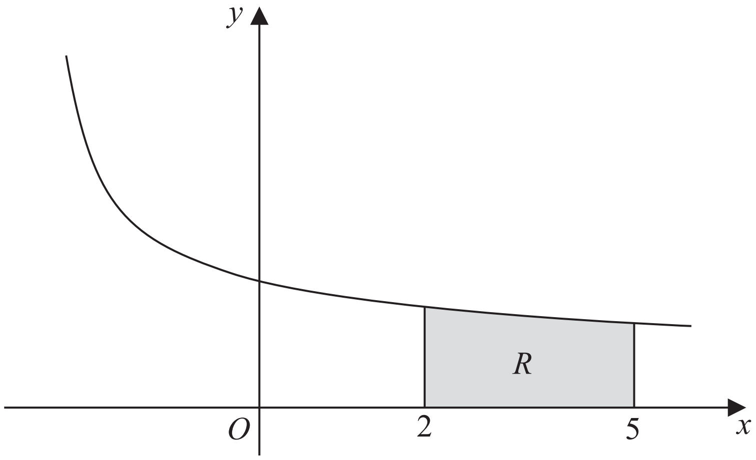

Diagram not drawn to scale

Figure 1

Figure 1 shows a sketch of part of the curve with equation $y = \dfrac{1}{\sqrt{2x+5}}$, $\quad x > -2.5$

The finite region R, shown shaded in Figure 1, is bounded by the curve, the x-axis and the lines with equations $x = 2$ and $x = 5$

(a) Use the trapezium rule with three strips of equal width to find an estimate for the area of R, giving your answer to 3 decimal places.

(4)

(b) Use calculus to find the exact area of R.

(4)

(c) Hence calculate the magnitude of the error of the estimate found in part (a), giving your answer to one significant figure.

(1)

Leave blank

5.

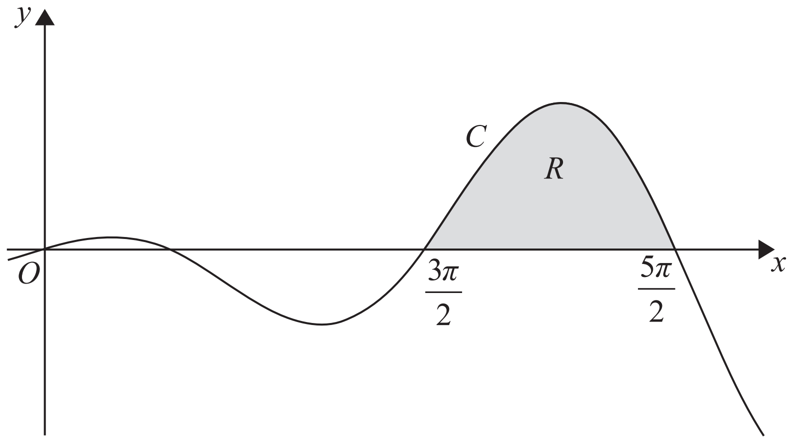

Figure 1

Figure 1 shows a sketch of part of the curve C with equation

$$y = x\cos x, \quad x \in \mathbb{R}$$

The finite region R, shown shaded in Figure 1, is bounded by the curve C and the x-axis for $\frac{3\pi}{2} \leqslant x \leqslant \frac{5\pi}{2}$

(a) Complete the table below with the exact value of y corresponding to $x = \frac{7\pi}{4}$ and with the exact value of y corresponding to $x = \frac{9\pi}{4}$

x	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π	$\frac{9\pi}{4}$	$\frac{5\pi}{2}$
y	0		2π		0

(1)

(b) Use the trapezium rule, with all five y values in the completed table, to find an approximate value for the area of R, giving your answer to 4 significant figures.

(3)

(c) Find

$$\int x\cos x\,\mathrm{d}x$$

(3)

(d) Using your answer from part (c), find the exact area of the region R.

(2)

Leave blank

8.

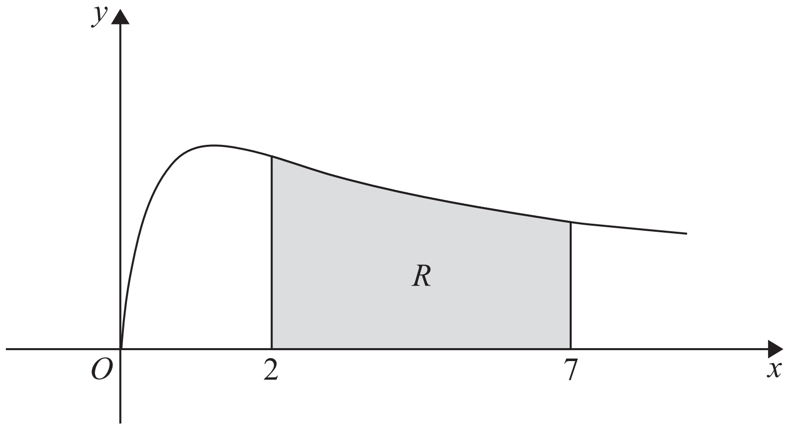

Figure 1

Figure 1 shows a sketch of part of the curve with equation $y = \sqrt{\frac{x}{x^2+1}}, \quad x \geqslant 0$

The finite region R, shown shaded in Figure 1, is bounded by the curve, the line with equation $x = 2$, the x-axis and the line with equation $x = 7$

The table below shows corresponding values of x and y for $y = \sqrt{\frac{x}{x^2+1}}$

x	2	3	4	5	6	7
y	0.6325	0.5477	0.4851	0.4385	0.4027	0.3742

(a) Use the trapezium rule, with all the values of y in the table, to find an estimate for the area of R, giving your answer to 3 decimal places.

(3)

The region R is rotated 360° about the x-axis to form a solid of revolution.

(b) Use calculus to find the exact volume of the solid of revolution formed. Write your answer in its simplest form.

(4)

Leave blank

6.

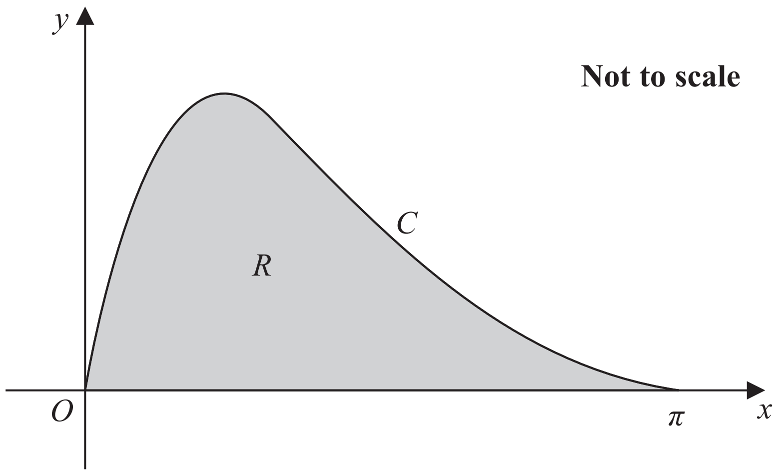

Figure 1

Figure 1 shows a sketch of the curve C with equation $y = 2e^{-x}\sqrt{\sin x}$, $0 \leqslant x \leqslant \pi$.
The finite region R, shown shaded in Figure 1, is bounded by the curve and the x-axis.

(a) Complete the table below with the value of y corresponding to $x = \frac{\pi}{2}$, giving your answer to 5 decimal places.

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
y	0	0.76679		0.15940	0

(1)

(b) Use the trapezium rule, with all the values of y in the completed table, to obtain an estimate for the area of the region R. Give your answer to 4 decimal places.

(3)

(c) Given $y = 2e^{-x}\sqrt{\sin x}$, find $\frac{dy}{dx}$ for $0 < x < \pi$.

(3)

The curve C has a maximum turning point when $x = a$.

(d) Use your answer to part (c) to find the value of a, giving your answer to 3 decimal places.

(3)

Leave blank

9. (a) Given that a is a constant, $a > 1$, sketch the graph of

$$y = a^x, \qquad x \in \mathbb{R}$$

On your diagram show the coordinates of the point where the graph crosses the y-axis.
(2)

The table below shows corresponding values of x and y for $y = 2^x$

x	–4	–2	0	2	4
y	0.0625	0.25	1	4	16

(b) Use the trapezium rule, with all of the values of y from the table, to find an approximate value, to 2 decimal places, for

$$\int_{-4}^{4} 2^x \, dx$$

(4)

(c) Use the answer to part (b) to find an approximate value for

(i) $\displaystyle\int_{-4}^{4} 2^{x+2} \, dx$

(ii) $\displaystyle\int_{-4}^{4} (3 + 2^x) \, dx$

(4)

DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA

10. The height above ground, H metres, of a passenger on a roller coaster can be modelled by the differential equation

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{H\cos(0.25t)}{40}$$

where t is the time, in seconds, from the start of the ride.

Given that the passenger is 5 m above the ground at the start of the ride,

(a) show that $H = 5\mathrm{e}^{0.1\sin(0.25t)}$

(5)

(b) State the maximum height of the passenger above the ground.

(1)

The passenger reaches the maximum height, for the second time, T seconds after the start of the ride.

(c) Find the value of T.

(2)

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

10. A spherical mint of radius 5 mm is placed in the mouth and sucked.
Four minutes later, the radius of the mint is 3 mm.

In a simple model, the rate of decrease of the radius of the mint is inversely proportional to the square of the radius.

Using this model and all the information given,

(a) find an equation linking the radius of the mint and the time.
(You should define the variables that you use.)
(5)

(b) Hence find the total time taken for the mint to completely dissolve. Give your answer in minutes and seconds to the nearest second.
(2)

(c) Suggest a limitation of the model.
(1)

14. A scientist is studying a population of mice on an island.

The number of mice, N, in the population, t months after the start of the study, is modelled by the equation

$$N = \frac{900}{3 + 7e^{-0.25t}}, \quad t \in \mathbb{R}, \quad t \geqslant 0$$

(a) Find the number of mice in the population at the start of the study.

(1)

(b) Show that the rate of growth $\frac{dN}{dt}$ is given by $\frac{dN}{dt} = \frac{N(300 - N)}{1200}$

(4)

The rate of growth is a maximum after T months.

(c) Find, according to the model, the value of T.

(4)

According to the model, the maximum number of mice on the island is P.

(d) State the value of P.

(1)

14. (a) Use the substitution $u = 4 - \sqrt{h}$ to show that

$$\int \frac{\mathrm{d}h}{4 - \sqrt{h}} = -8\ln\left|4 - \sqrt{h}\right| - 2\sqrt{h} + k$$

where k is a constant

(6)

A team of scientists is studying a species of slow growing tree.

The rate of change in height of a tree in this species is modelled by the differential equation

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{t^{0.25}\left(4 - \sqrt{h}\right)}{20}$$

where h is the height in metres and t is the time, measured in years, after the tree is planted.

(b) Find, according to the model, the range in heights of trees in this species.

(2)

One of these trees is one metre high when it is first planted.

According to the model,

(c) calculate the time this tree would take to reach a height of 12 metres, giving your answer to 3 significant figures.

(7)

8. A new smartphone was released by a company.

The company monitored the total number of phones sold, n, at time t days after the phone was released.

The company observed that, during this time,

the rate of increase of n was proportional to n

Use this information to write down a suitable equation for n in terms of t.

(*You do not need to evaluate any unknown constants in your equation*.)

(2)

14. A large spherical balloon is deflating.

At time t seconds the balloon has radius r cm and volume V cm^3

The volume of the balloon is modelled as decreasing at a constant rate.

(a) Using this model, show that

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -\frac{k}{r^2}$$

where k is a positive constant.

(3)

Given that

- the initial radius of the balloon is 40 cm
- after 5 seconds the radius of the balloon is 20 cm
- the volume of the balloon continues to decrease at a constant rate until the balloon is empty

(b) solve the differential equation to find a complete equation linking r and t.

(5)

(c) Find the limitation on the values of t for which the equation in part (b) is valid.

(2)

Leave blank

9. (a) Use the substitution $u = 4 - \sqrt{x}$ to find

$$\int \frac{dx}{4 - \sqrt{x}}$$

(6)

A team of scientists is studying a species of slow growing tree.

The rate of change in height of a tree in this species is modelled by the differential equation

$$\frac{dh}{dt} = \frac{4 - \sqrt{h}}{20}$$

where h is the height in metres and t is the time measured in years after the tree is planted.

(b) Find the range in values of h for which the height of a tree in this species is increasing.

(2)

(c) Given that one of these trees is 1 metre high when it is planted, calculate the time it would take to reach a height of 10 metres. Write your answer to 3 significant figures.

(7)

Leave blank

6. (a) Express $\dfrac{5-4x}{(2x-1)(x+1)}$ in partial fractions.

(3)

(b) (i) Find a general solution of the differential equation

$$(2x-1)(x+1)\frac{dy}{dx} = (5-4x)y, \quad x > \frac{1}{2}$$

Given that $y = 4$ when $x = 2$,

(ii) find the particular solution of this differential equation.
Give your answer in the form $y = f(x)$.

(7)

Leave blank

8. (a) Prove by differentiation that

$$\frac{\mathrm{d}}{\mathrm{d}y}(\ln \tan 2y) = \frac{4}{\sin 4y}, \qquad 0 < y < \frac{\pi}{4}$$

(4)

(b) Given that $y = \frac{\pi}{6}$ when $x = 0$, solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2\cos x \sin 4y, \qquad 0 < y < \frac{\pi}{4}$$

Give your answer in the form $\tan 2y = A\mathrm{e}^{B\sin x}$, where A and B are constants to be determined.

(6)

Leave blank

9. (a) Express $\dfrac{3x^2 - 4}{x^2(3x - 2)}$ in partial fractions.

(4)

(b) Given that $x > \dfrac{2}{3}$, find the general solution of the differential equation

$$x^2(3x - 2)\,\frac{dy}{dx} = y(3x^2 - 4)$$

Give your answer in the form $y = f(x)$.

(6)

Leave blank

13.

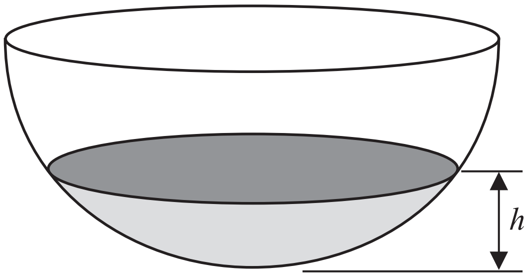

Figure 4

Figure 4 shows a hemispherical bowl containing some water.

At t seconds, the height of the water is h cm and the volume of the water is V cm^3, where

$$V = \frac{1}{3}\pi h^2(30 - h), \qquad 0 < h \leqslant 10$$

The water is leaking from a hole in the bottom of the bowl.

Given that $\frac{\mathrm{d}V}{\mathrm{d}t} = -\frac{1}{10}V$

(a) show that $\frac{\mathrm{d}h}{\mathrm{d}t} = -\frac{h(30-h)}{30(20-h)}$

(5)

(b) Write $\frac{30(20-h)}{h(30-h)}$ in partial fraction form.

(3)

Given that $h = 10$ when $t = 0$,

(c) use your answers to parts (a) and (b) to find the time taken for the height of the water to fall to 5 cm. Give your answer in seconds to 2 decimal places.

(6)

Leave blank

12. In freezing temperatures, ice forms on the surface of the water in a barrel. At time t hours after the start of freezing, the thickness of the ice formed is x mm. You may assume that the thickness of the ice is uniform across the surface of the water.

At 4pm there is no ice on the surface, and freezing begins.

At 6pm, after two hours of freezing, the ice is 1.5 mm thick.

In a simple model, the rate of increase of x, in mm per hour, is assumed to be constant for a period of 20 hours.

Using this simple model,

(a) express t in terms of x,

(2)

(b) find the value of t when $x = 3$

(1)

In a second model, the rate of increase of x, in mm per hour, is given by the differential equation

$$\frac{dx}{dt} = \frac{\lambda}{(2x+1)} \quad \text{where } \lambda \text{ is a constant and } 0 \leqslant t \leqslant 20$$

Using this second model,

(c) solve the differential equation and express t in terms of x and λ,

(3)

(d) find the exact value for λ,

(1)

(e) find at what time the ice is predicted to be 3 mm thick.

(2)

Leave blank

12.

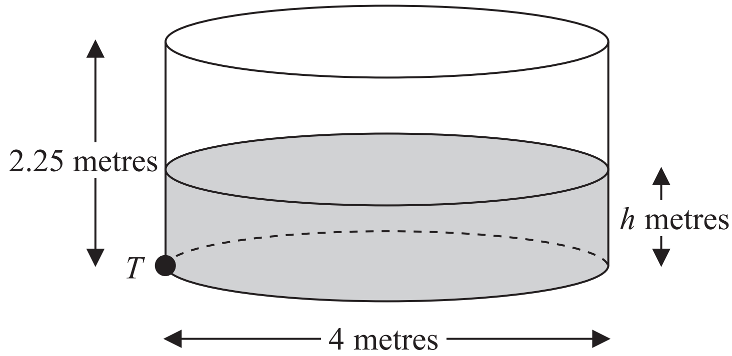

Figure 4

Figure 4 shows a right cylindrical water tank. The diameter of the circular cross section of the tank is 4 m and the height is 2.25 m. Water is flowing into the tank at a constant rate of 0.4π m^3 min^{-1}. There is a tap at a point T at the base of the tank. When the tap is open, water leaves the tank at a rate of $0.2\pi\sqrt{h}$ m^3 min^{-1}, where h is the height of the water in metres.

(a) Show that at time t minutes after the tap has been opened, the height h m of the water in the tank satisfies the differential equation

$$20\frac{\mathrm{d}h}{\mathrm{d}t} = 2 - \sqrt{h}$$

(5)

At the instant when the tap is opened, $t = 0$ and $h = 0.16$

(b) Use the differential equation to show that the time taken to fill the tank to a height of 2.25 m is given by

$$\int_{0.16}^{2.25} \frac{20}{2 - \sqrt{h}}\,\mathrm{d}h$$

(2)

Using the substitution $h = (2 - x)^2$, or otherwise,

(c) find the time taken to fill the tank to a height of 2.25 m.

Give your answer in minutes to the nearest minute.

(*Solutions based entirely on graphical or numerical methods are not acceptable.*)

(7)

Leave blank

11. (a) Given $0 \leqslant h < 25$, use the substitution $u = 5 - \sqrt{h}$ to show that

$$\int \frac{\mathrm{d}h}{5 - \sqrt{h}} = -10\ln\left(5 - \sqrt{h}\right) - 2\sqrt{h} + k$$

where k is a constant.

(6)

A team of scientists is studying a species of tree.

The rate of change in height of a tree of this species is modelled by the differential equation

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{t^{0.2}\left(5 - \sqrt{h}\right)}{5}$$

where h is the height of the tree in metres and t is the time in years after the tree is planted.

One of these trees is 2 metres high when it is planted.

(b) Use integration to calculate the time it would take for this tree to reach a height of 15 metres, giving your answer to one decimal place.

(7)

(c) Hence calculate the rate of change in height of this tree when its height is 15 metres. Write your answer in centimetres per year to the nearest centimetre.

(1)

Leave blank

7.

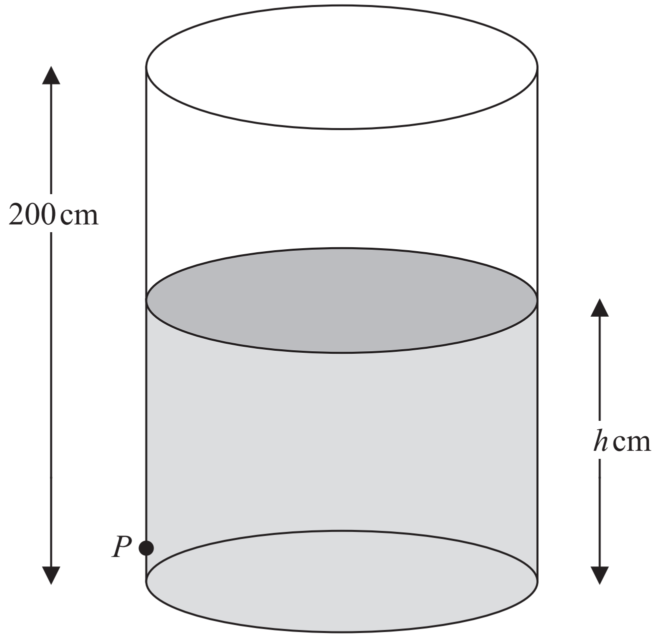

Figure 3

Figure 3 shows a vertical cylindrical tank of height 200 cm containing water.
Water is leaking from a hole P on the side of the tank.

At time t minutes after the leaking starts, the height of water in the tank is h cm.

The height h cm of the water in the tank satisfies the differential equation

$$\frac{\mathrm{d}h}{\mathrm{d}t} = k(h-9)^{\frac{1}{2}}, \qquad 9 < h \leqslant 200$$

where k is a constant.

Given that, when $h = 130$, the height of the water is falling at a rate of 1.1 cm per minute,

(a) find the value of k.

(2)

Given that the tank was full of water when the leaking started,

(b) solve the differential equation with your value of k, to find the value of t when $h = 50$

(6)

Leave blank

7. (a) Express $\dfrac{2}{P(P-2)}$ in partial fractions.

(3)

A team of biologists is studying a population of a particular species of animal.

The population is modelled by the differential equation

$$\frac{dP}{dt} = \frac{1}{2}P(P-2)\cos 2t, \quad t \geqslant 0$$

where P is the population in thousands, and t is the time measured in years since the start of the study.

Given that $P = 3$ when $t = 0$,

(b) solve this differential equation to show that

$$P = \frac{6}{3 - e^{\frac{1}{2}\sin 2t}}$$

(7)

(c) find the time taken for the population to reach 4000 for the first time.
Give your answer in years to 3 significant figures.

(3)

Leave blank

8. In an experiment testing solid rocket fuel, some fuel is burned and the waste products are collected. Throughout the experiment the sum of the masses of the unburned fuel and waste products remains constant.

Let x be the mass of waste products, in kg, at time t minutes after the start of the experiment. It is known that at time t minutes, the rate of increase of the mass of waste products, in kg per minute, is k times the mass of unburned fuel remaining, where k is a positive constant.

The differential equation connecting x and t may be written in the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = k(M - x), \text{ where } M \text{ is a constant.}$$

(a) Explain, in the context of the problem, what $\frac{\mathrm{d}x}{\mathrm{d}t}$ and M represent.
(2)

Given that initially the mass of waste products is zero,

(b) solve the differential equation, expressing x in terms of k, M and t.
(6)

Given also that $x = \frac{1}{2}M$ when $t = \ln 4$,

(c) find the value of x when $t = \ln 9$, expressing x in terms of M, in its simplest form.
(4)