9. A company plans to build a new fairground ride. The ride will consist of a capsule that will hold the passengers and the capsule will be attached to a tall tower. The capsule is to be released from rest from a point half way up the tower and then made to oscillate in a vertical line.

The vertical displacement, x metres, of the top of the capsule below its initial position at time t seconds is modelled by the differential equation,

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4\frac{\mathrm{d}x}{\mathrm{d}t} + x = 200\cos t, \qquad t \geqslant 0$$

where m is the mass of the capsule including its passengers, in thousands of kilograms.

The maximum permissible weight for the capsule, including its passengers, is 30 000 N.

Taking the value of g to be $10\,\mathrm{ms}^{-2}$ and assuming the capsule is at its maximum permissible weight,

(a) (i) explain why the value of m is 3

(ii) show that a particular solution to the differential equation is

$$x = 40\sin t - 20\cos t$$

(iii) hence find the general solution of the differential equation.

(8)

(b) Using the model, find, to the nearest metre, the vertical distance of the top of the capsule from its initial position, 9 seconds after it is released.

(4)

7. At the start of the year 2000, a survey began of the number of foxes and rabbits on an island.

At time t years after the survey began, the number of foxes, f, and the number of rabbits, r, on the island are modelled by the differential equations

$$\frac{\mathrm{d}f}{\mathrm{d}t} = 0.2f + 0.1r$$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -0.2f + 0.4r$$

(a) Show that $\frac{\mathrm{d}^2 f}{\mathrm{d}t^2} - 0.6\frac{\mathrm{d}f}{\mathrm{d}t} + 0.1f = 0$

(3)

(b) Find a general solution for the number of foxes on the island at time t years.

(4)

(c) Hence find a general solution for the number of rabbits on the island at time t years.

(3)

At the start of the year 2000 there were 6 foxes and 20 rabbits on the island.

(d) (i) According to this model, in which year are the rabbits predicted to die out?

(ii) According to this model, how many foxes will be on the island when the rabbits die out?

(iii) Use your answers to parts (i) and (ii) to comment on the model.

(7)

8. A scientist is studying the effect of introducing a population of white-clawed crayfish into a population of signal crayfish.
At time t years, the number of white-clawed crayfish, w, and the number of signal crayfish, s, are modelled by the differential equations

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{5}{2}(w - s)$$
$$\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{2}{5}w - 90\mathrm{e}^{-t}$$

(a) Show that

$$2\frac{\mathrm{d}^2w}{\mathrm{d}t^2} - 5\frac{\mathrm{d}w}{\mathrm{d}t} + 2w = 450\mathrm{e}^{-t}$$

(3)

(b) Find a general solution for the number of white-clawed crayfish at time t years.

(6)

(c) Find a general solution for the number of signal crayfish at time t years.

(2)

The model predicts that, at time T years, the population of white-clawed crayfish will have died out.

Given that $w = 65$ and $s = 85$ when $t = 0$

(d) find the value of T, giving your answer to 3 decimal places.

(6)

(e) Suggest a limitation of the model.

(1)

5. An engineer is investigating the motion of a sprung diving board at a swimming pool.
Let E be the position of the end of the diving board when it is at rest in its equilibrium position and when there is no diver standing on the diving board.
A diver jumps from the diving board.
The vertical displacement, h cm, of the end of the diving board above E is modelled by the differential equation

$$4\frac{d^2h}{dt^2} + 4\frac{dh}{dt} + 37h = 0$$

where t seconds is the time after the diver jumps.

(a) Find a general solution of the differential equation. **(2)**

When $t = 0$, the end of the diving board is 20 cm below E and is moving upwards with a speed of 55 cm s^{-1}.

(b) Find, according to the model, the maximum vertical displacement of the end of the diving board above E. **(8)**

(c) Comment on the suitability of the model for large values of t. **(2)**

5. Two compounds, X and Y, are involved in a chemical reaction. The amounts in grams of these compounds, t minutes after the reaction starts, are x and y respectively and are modelled by the differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -5x + 10y - 30$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -2x + 3y - 4$$

(a) Show that

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\frac{\mathrm{d}x}{\mathrm{d}t} + 5x = 50$$

(3)

(b) Find, according to the model, a general solution for the amount in grams of compound X present at time t minutes.

(6)

(c) Find, according to the model, a general solution for the amount in grams of compound Y present at time t minutes.

(3)

Given that $x = 2$ and $y = 5$ when $t = 0$

(d) find

(i) the particular solution for x,

(ii) the particular solution for y.

(4)

A scientist thinks that the chemical reaction will have stopped after 8 minutes.

(e) Explain whether this is supported by the model.

(1)

3. A scientist is investigating the concentration of antibodies in the bloodstream of a patient following a vaccination.
The concentration of antibodies, x, measured in micrograms (μg) per millilitre (ml) of blood, is modelled by the differential equation

$$100\frac{d^2x}{dt^2} + 60\frac{dx}{dt} + 13x = 26$$

where t is the number of weeks since the vaccination was given.

(a) Find a general solution of the differential equation.

(4)

Initially,
- there are no antibodies in the bloodstream of the patient
- the concentration of antibodies is estimated to be increasing at 10 μg/ml per week

(b) Find, according to the model, the maximum concentration of antibodies in the bloodstream of the patient after the vaccination.

(8)

A second dose of the vaccine has to be given to try to ensure that it is fully effective.
It is only safe to give the second dose if the concentration of antibodies in the bloodstream of the patient is less than 5 μg/ml.

(c) Determine whether, according to the model, it is safe to give the second dose of the vaccine to the patient exactly 10 weeks after the first dose.

(2)

6. A tourist decides to do a bungee jump from a bridge over a river.
One end of an elastic rope is attached to the bridge and the other end of the elastic rope is attached to the tourist.
The tourist jumps off the bridge.

At time t seconds after the tourist reaches their lowest point, their vertical displacement is x metres above a fixed point 30 metres vertically above the river.

When $t = 0$

- $x = -20$
- the velocity of the tourist is $0\,\mathrm{m\,s^{-1}}$
- the acceleration of the tourist is $13.6\,\mathrm{m\,s^{-2}}$

In the subsequent motion, the elastic rope is assumed to remain taut so that the vertical displacement of the tourist can be modelled by the differential equation

$$5k\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2k\frac{\mathrm{d}x}{\mathrm{d}t} + 17x = 0 \qquad t \geqslant 0$$

where k is a positive constant.

(a) Determine the value of k

(2)

(b) Determine the particular solution to the differential equation.

(7)

(c) Hence find, according to the model, the vertical height of the tourist above the river 15 seconds after they have reached their lowest point.

(2)

(d) Give a limitation of the model.

(1)

10.

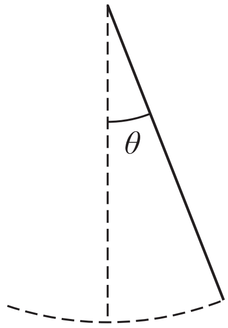

Figure 3

The motion of a pendulum, shown in Figure 3, is modelled by the differential equation

$$\frac{d^2\theta}{dt^2} + 9\theta = \frac{1}{2}\cos 3t$$

where θ is the angle, in radians, that the pendulum makes with the downward vertical, t seconds after it begins to move.

(a) (i) Show that a particular solution of the differential equation is

$$\theta = \frac{1}{12}t\sin 3t$$

(4)

(ii) Hence, find the general solution of the differential equation.

(4)

Initially, the pendulum

- makes an angle of $\frac{\pi}{3}$ radians with the downward vertical
- is at rest

Given that, 10 seconds after it begins to move, the pendulum makes an angle of α radians with the downward vertical,

(b) determine, according to the model, the value of α to 3 significant figures.

(4)

Given that the true value of α is 0.62

(c) evaluate the model.

(1)

The differential equation

$$\frac{d^2\theta}{dt^2} + 9\theta = \frac{1}{2}\cos 3t$$

models the motion of the pendulum as moving with forced harmonic motion.

(d) Refine the differential equation so that the motion of the pendulum is simple harmonic motion.

(1)

9. A patient is treated by administering an antibiotic intravenously at a constant rate for some time.

Initially there is none of the antibiotic in the patient.

At time t minutes after treatment began

- the concentration of the antibiotic in the blood of the patient is x mg/ml
- the concentration of the antibiotic in the tissue of the patient is y mg/ml

The concentration of antibiotic in the patient is modelled by the equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 0.025y - 0.045x + 2$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 0.032x - 0.025y$$

(a) Show that

$$40\,000\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 2800\frac{\mathrm{d}y}{\mathrm{d}t} + 13y = 2560$$

(3)

(b) Determine, according to the model, a general solution for the concentration of the antibiotic in the patient's tissue at time t minutes after treatment began.

(5)

(c) Hence determine a particular solution for the concentration of the antibiotic in the tissue at time t minutes after treatment began.

(4)

To be effective for the patient the concentration of antibiotic in the tissue must eventually reach a level between 185 mg/ml and 200 mg/ml.

(d) Determine whether the rate of administration of the antibiotic is effective for the patient, giving a reason for your answer.

(2)