Answer ALL questions. Write your answers in the spaces provided.

1. Prove that

$$\sum_{r=1}^{n} \frac{1}{(r+1)(r+3)} = \frac{n(an+b)}{12(n+2)(n+3)}$$

where a and b are constants to be found.

(5)

4. Prove that, for $n \in \mathbb{Z}, n \geqslant 0$

$$\sum_{r=0}^{n} \frac{1}{(r+1)(r+2)(r+3)} = \frac{(n+a)(n+b)}{c(n+2)(n+3)}$$

where a, b and c are integers to be found.

(5)

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4. In this question you may assume the results for

$$\sum_{r=1}^{n} r^3 \ , \ \sum_{r=1}^{n} r^2 \ \text{ and } \ \sum_{r=1}^{n} r$$

(a) Show that the sum of the cubes of the first n positive odd numbers is

$$n^2(2n^2 - 1)$$

(5)

The sum of the cubes of 10 consecutive positive odd numbers is 99 800

(b) Use the answer to part (a) to determine the smallest of these 10 consecutive positive odd numbers.

(4)

4. (a) Use the method of differences to prove that for $n > 2$

$$\sum_{r=2}^{n} \ln\left(\frac{r+1}{r-1}\right) \equiv \ln\left(\frac{n(n+1)}{2}\right)$$

(4)

(b) Hence find the exact value of

$$\sum_{r=51}^{100} \ln\left(\frac{r+1}{r-1}\right)^{35}$$

Give your answer in the form $a\ln\left(\frac{b}{c}\right)$ where a, b and c are integers to be determined.

(3)

7. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

(a) Explain why, for $n \in \mathbb{N}$

$$\sum_{r=1}^{2n} (-1)^r \mathrm{f}(r) = \sum_{r=1}^{n} \left(\mathrm{f}(2r) - \mathrm{f}(2r-1)\right)$$

for any function $\mathrm{f}(r)$.

(2)

(b) Use the standard summation formulae to show that, for $n \in \mathbb{N}$

$$\sum_{r=1}^{2n} r\left((-1)^r + 2r\right)^2 = n(2n+1)(8n^2 + 4n + 5)$$

(6)

(c) Hence evaluate

$$\sum_{r=14}^{50} r\left((-1)^r + 2r\right)^2$$

(4)

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6. (a) Use the standard results for $\sum_{r=1}^{n} r^2$ and $\sum_{r=1}^{n} r$ to show that

$$\sum_{r=1}^{n}(3r-2)^2 = \frac{1}{2}n\left[6n^2 - 3n - 1\right]$$

for all positive integers n.

(5)

(b) Hence find any values of n for which

$$\sum_{r=5}^{n}(3r-2)^2 + 103\sum_{r=1}^{28} r\cos\left(\frac{r\pi}{2}\right) = 3n^3$$

(5)

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6. An art display consists of an arrangement of n marbles.

When arranged in ascending order of mass, the mass of the first marble is 10 grams.
The mass of each subsequent marble is 3 grams more than the mass of the previous one, so that the rth marble has mass $(7 + 3r)$ grams.

(a) Show that the mean mass, in grams, of the marbles in the display is given by

$$\frac{1}{2}(3n+17)$$

(3)

Given that there are 85 marbles in the display,

(b) use the standard summation formulae to find the standard deviation of the mass of the marbles in the display, giving your answer, in grams, to one decimal place.

(6)

5.

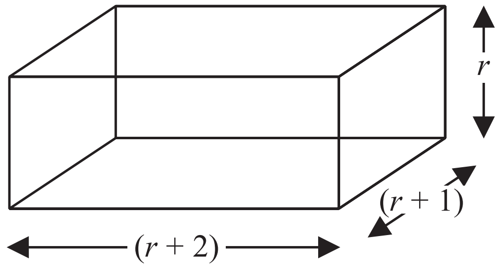

Figure 2

A block has length $(r + 2)$ cm, width $(r + 1)$ cm and height r cm, as shown in Figure 2.

In a set of n such blocks, the first block has a height of 1 cm, the second block has a height of 2 cm, the third block has a height of 3 cm and so on.

(a) Use the standard results for $\sum_{r=1}^{n} r^3, \sum_{r=1}^{n} r^2$ and $\sum_{r=1}^{n} r$ to show that the **total** volume, V, of all n blocks in the set is given by

$$V = \frac{n}{4}(n+1)(n+2)(n+3) \qquad n \geqslant 1$$

(5)

Given that the total volume of all n blocks is

$$(n^4 + 6n^3 - 11\,710)\ \text{cm}^3$$

(b) determine how many blocks make up the set.

(2)

3. (a) Use the standard results for summations to show that for all positive integers n

$$\sum_{r=1}^{n} (5r-2)^2 = \frac{1}{6}n(an^2 + bn + c)$$

where a, b and c are integers to be determined.

(5)

(b) Hence determine the value of k for which

$$\sum_{r=1}^{k} (5r-2)^2 = 94k^2$$

(4)

5. (a) Use the standard summation formulae to show that, for $n \in \mathbb{N}$,

$$\sum_{r=1}^{n}\left(3r^2 - 17r - 25\right) = n\left(n^2 - An - B\right)$$

where A and B are integers to be determined.

(4)

(b) Explain why, for $k \in \mathbb{N}$,

$$\sum_{r=1}^{3k} r\tan\left(60r\right)^\circ = -k\sqrt{3}$$

(2)

Using the results from part (a) and part (b) and showing all your working,

(c) determine any value of n that satisfies

$$\sum_{r=5}^{n}\left(3r^2 - 17r - 25\right) = 15\left[\sum_{r=6}^{3n} r\tan\left(60r\right)^\circ\right]^2$$

(6)

8. (a) Use the standard results for $\sum_{r=1}^{n} r^2$ and $\sum_{r=1}^{n} r$ to show that, for all positive integers n,

$$\sum_{r=1}^{n} (2r-1)^2 = \frac{n}{3}(an^2-1)$$

where a is a constant to be determined.

(5)

(b) Hence determine the sum of the squares of all positive odd three-digit integers.

(3)