4. A complex number z has modulus 1 and argument θ.

(a) Show that

$$z^n + \frac{1}{z^n} = 2\cos n\theta, \qquad n \in \mathbb{Z}^+$$

(2)

(b) Hence, show that

$$\cos^4 \theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$$

(5)

6. (a) (i) Show on an Argand diagram the locus of points given by the values of z satisfying

$$|z - 4 - 3\mathbf{i}| = 5$$

Taking the initial line as the positive real axis with the pole at the origin and given that $\theta \in [\alpha, \alpha + \pi]$, where $\alpha = -\arctan\left(\frac{4}{3}\right)$,

(ii) show that this locus of points can be represented by the polar curve with equation

$$r = 8\cos\theta + 6\sin\theta$$

(6)

The set of points A is defined by

$$A = \left\{z : 0 \leqslant \arg z \leqslant \frac{\pi}{3}\right\} \cap \{z : |z - 4 - 3\mathbf{i}| \leqslant 5\}$$

(b) (i) Show, by shading on your Argand diagram, the set of points A.

(ii) Find the **exact** area of the region defined by A, giving your answer in simplest form.

(7)

4. The infinite series C and S are defined by

$$C = \cos\theta + \frac{1}{2}\cos 5\theta + \frac{1}{4}\cos 9\theta + \frac{1}{8}\cos 13\theta + \ldots$$

$$S = \sin\theta + \frac{1}{2}\sin 5\theta + \frac{1}{4}\sin 9\theta + \frac{1}{8}\sin 13\theta + \ldots$$

Given that the series C and S are both convergent,

(a) show that

$$C + iS = \frac{2e^{i\theta}}{2 - e^{4i\theta}}$$

(4)

(b) Hence show that

$$S = \frac{4\sin\theta + 2\sin 3\theta}{5 - 4\cos 4\theta}$$

(4)

6. In an Argand diagram, the points A, B and C are the vertices of an equilateral triangle with its centre at the origin. The point A represents the complex number $6 + 2\text{i}$.

(a) Find the complex numbers represented by the points B and C, giving your answers in the form $x + \text{i}y$, where x and y are real and exact.

(6)

The points D, E and F are the midpoints of the sides of triangle ABC.

(b) Find the exact area of triangle DEF.

(3)

2. In an Argand diagram, the points A and B are represented by the complex numbers $-3 + 2\text{i}$ and $5 - 4\text{i}$ respectively. The points A and B are the end points of a diameter of a circle C.

(a) Find the equation of C, giving your answer in the form

$$|z - a| = b \qquad a \in \mathbb{C},\ b \in \mathbb{R}$$

(3)

The circle D, with equation $|z - 2 - 3\text{i}| = 2$, intersects C at the points representing the complex numbers z_1 and z_2

(b) Find the complex numbers z_1 and z_2

(6)

4. (a) Use de Moivre's theorem to prove that

$$\sin 7\theta = 7\sin\theta - 56\sin^3\theta + 112\sin^5\theta - 64\sin^7\theta$$

(5)

(b) Hence find the distinct roots of the equation

$$1 + 7x - 56x^3 + 112x^5 - 64x^7 = 0$$

(5)

giving your answer to 3 decimal places where appropriate.

1. Given that

$$z_1 = 3\left(\cos\left(\frac{\pi}{3}\right) + \mathrm{i}\sin\left(\frac{\pi}{3}\right)\right)$$

$$z_2 = \sqrt{2}\left(\cos\left(\frac{\pi}{12}\right) - \mathrm{i}\sin\left(\frac{\pi}{12}\right)\right)$$

(a) write down the exact value of

(i) $|z_1 z_2|$

(ii) $\arg(z_1 z_2)$

(2)

Given that $w = z_1 z_2$ and that $\arg(w^n) = 0$, where $n \in \mathbb{Z}^+$

(b) determine

(i) the smallest positive value of n

(ii) the corresponding value of $|w^n|$

(3)

8. (i) The point P is one vertex of a regular pentagon in an Argand diagram. The centre of the pentagon is at the origin.

Given that P represents the complex number $6 + 6\mathrm{i}$, determine the complex numbers that represent the other vertices of the pentagon, giving your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$

(5)

(ii) (a) On a single Argand diagram, shade the region, R, that satisfies both

$$|z - 2\mathrm{i}| \leqslant 2 \qquad \text{and} \qquad \frac{1}{4}\pi \leqslant \arg z \leqslant \frac{1}{3}\pi$$

(2)

(b) Determine the exact area of R, giving your answer in simplest form.

(4)

9. (a) Given that $|z| < 1$, write down the sum of the infinite series

$$1 + z + z^2 + z^3 + \dots$$

(1)

(b) Given that $z = \frac{1}{2}(\cos\theta + \mathrm{i}\sin\theta)$,

(i) use the answer to part (a), and de Moivre's theorem or otherwise, to prove that

$$\frac{1}{2}\sin\theta + \frac{1}{4}\sin 2\theta + \frac{1}{8}\sin 3\theta + \dots \quad = \frac{2\sin\theta}{5 - 4\cos\theta}$$

(5)

(ii) show that the sum of the infinite series $1 + z + z^2 + z^3 + \dots$ cannot be purely imaginary, giving a reason for your answer.

(2)

1. $\mathrm{f}(z) = z^3 + az + 52$ where a is a real constant

Given that $2 - 3\mathrm{i}$ is a root of the equation $\mathrm{f}(z) = 0$

(a) write down the other complex root.

(1)

(b) Hence

(i) solve completely $\mathrm{f}(z) = 0$

(ii) determine the value of a

(4)

(c) Show all the roots of the equation $\mathrm{f}(z) = 0$ on a single Argand diagram.

(1)

7. Given that $z = a + b\text{i}$ is a complex number where a and b are real constants,

(a) show that zz^* is a real number.

(2)

Given that

- $zz^* = 18$
- $\dfrac{z}{z^*} = \dfrac{7}{9} + \dfrac{4\sqrt{2}}{9}\text{i}$

(b) determine the possible complex numbers z

(5)

1. A student was asked to answer the following:

For the complex numbers $z_1 = 3 - 3\text{i}$ and $z_2 = \sqrt{3} + \text{i}$, find the value of $\arg\left(\dfrac{z_1}{z_2}\right)$

The student's attempt is shown below.

Line 1 → $\arg(z_1) = \tan^{-1}\left(\dfrac{3}{3}\right) = \dfrac{\pi}{4}$

Line 2 → $\arg(z_2) = \tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right) = \dfrac{\pi}{6}$

Line 3 → $\arg\left(\dfrac{z_1}{z_2}\right) = \dfrac{\arg(z_1)}{\arg(z_2)}$

Line 4 → $= \left(\dfrac{\pi}{4}\right) \Big/ \left(\dfrac{\pi}{6}\right) = \dfrac{3}{2}$

The student made errors in line 1 and line 3

Correct the error that the student made in

(a) (i) line 1

(ii) line 3

(2)

(b) Write down the correct value of $\arg\left(\dfrac{z_1}{z_2}\right)$

(1)

4. (i) Given that

$$z_1 = 6e^{\frac{\pi}{3}i} \text{ and } z_2 = 6\sqrt{3}e^{\frac{5\pi}{6}i}$$

show that

$$z_1 + z_2 = 12e^{\frac{2\pi}{3}i}$$

(3)

(ii) Given that

$$\arg(z - 5) = \frac{2\pi}{3}$$

determine the least value of $|z|$ as z varies.

(3)

3. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

$$z_1 = -4 + 4\text{i}$$

(a) Express z_1 in the form $r(\cos\theta + \text{i}\sin\theta)$, where $r \in \mathbb{R}$, $r > 0$ and $0 \leqslant \theta < 2\pi$

(2)

$$z_2 = 3\left(\cos\frac{17\pi}{12} + \text{i}\sin\frac{17\pi}{12}\right)$$

(b) Determine in the form $a + \text{i}b$, where a and b are exact real numbers,

(i) $\dfrac{z_1}{z_2}$

(2)

(ii) $(z_2)^4$

(2)

(c) Show on a single Argand diagram

(i) the complex numbers z_1, z_2 and $\dfrac{z_1}{z_2}$

(ii) the region defined by $\{z \in \mathbb{C} : |z - z_1| < |z - z_2|\}$

(4)

5. The points representing the complex numbers $z_1 = 35 - 25\mathrm{i}$ and $z_2 = -29 + 39\mathrm{i}$ are opposite vertices of a regular hexagon, H, in the complex plane.

The centre of H represents the complex number α

(a) Show that $\alpha = 3 + 7\mathrm{i}$

(2)

Given that $\beta = \dfrac{1 + \mathrm{i}}{64}$

(b) show that

$$\beta(z_1 - \alpha) = 1$$

(2)

The vertices of H are given by the roots of the equation

$$(\beta(z - \alpha))^6 = 1$$

(c) (i) Write down the roots of the equation $w^6 = 1$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$

(1)

(ii) Hence, or otherwise, determine the position of the other four vertices of H, giving your answers as complex numbers in Cartesian form.

(4)

3. (a) Shade on an Argand diagram the set of points

$$\{z \in \mathbb{C} : |z - 1 - \mathrm{i}| \leqslant 3\} \cap \left\{z \in \mathbb{C} : \frac{\pi}{4} \leqslant \arg(z - 2) \leqslant \frac{3\pi}{4}\right\}$$

(5)

The complex number w satisfies

$$|w - 1 - \mathrm{i}| = 3 \text{ and } \arg(w - 2) = \frac{\pi}{4}$$

(b) Find, in simplest form, the exact value of $|w|^2$

(4)

DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA

7.

$$f(z) = z^3 + z^2 + pz + q$$

where p and q are real constants.

The equation $f(z) = 0$ has roots z_1, z_2 and z_3
When plotted on an Argand diagram, the points representing z_1, z_2 and z_3 form the vertices of a triangle of area 35

Given that $z_1 = 3$, find the values of p and q.

(7)