AL Further Core Maths Paper 2 (9FM0/02) 1906

Mark Scheme – Pre-Stand

Question	Scheme	Marks	AOs
1(a)	$y=\tanh^{-1}(x)\Rightarrow \tanh y=x\Rightarrow x=\dfrac{\sinh y}{\cosh y}=\dfrac{e^{y}-e^{-y}}{e^{y}+e^{-y}}$	M1 A1	2.1 1.1b
	Note that some candidates only have one variable and reach e.g. $x=\dfrac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$ or $\tanh x=\dfrac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$ Allow this to score M1A1		
	$x\left(e^{2y}+1\right)=e^{2y}-1\Rightarrow e^{2y}(1-x)=1+x\Rightarrow e^{2y}=\dfrac{1+x}{1-x}$	M1	1.1b
	$e^{2y}=\dfrac{1+x}{1-x}\Rightarrow 2y=\ln\left(\dfrac{1+x}{1-x}\right)\Rightarrow y=\dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$ *	A1*	2.1
	Note that $e^{2y}(x-1)+x+1=0$ can be solved as a quadratic in e^{y}: $e^{y}=\dfrac{-\sqrt{0-4(x-1)(x+1)}}{2(x-1)}=\dfrac{-\sqrt{4(1-x)(x+1)}}{2(x-1)}=\dfrac{2\sqrt{(1-x)(x+1)}}{2(1-x)}$ $=\dfrac{\sqrt{(x+1)}}{\sqrt{(1-x)}}\Rightarrow y=\dfrac{1}{2}\ln\dfrac{(x+1)}{(1-x)}$ * Score **M1** for an attempt at the quadratic formula to make e^{y} the subject (condone $\pm\sqrt{\ldots}$) and **A1*** for a correct solution that rejects the positive root at some point and deals with the $(x-1)$ bracket correctly		
	$k=1$ or $-1<x<1$	B1	1.1b
		(5)	
(a) **Way 2**	$\tanh^{-1}x=\dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)\Rightarrow x=\tanh\left(\dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)\right)=\dfrac{e^{\ln\frac{1+x}{1-x}}-1}{e^{\ln\frac{1+x}{1-x}}+1}$	M1 A1	2.1 1.1b
	$x=\dfrac{e^{\ln\frac{1+x}{1-x}}-1}{e^{\ln\frac{1+x}{1-x}}+1}=\dfrac{\frac{1+x}{1-x}-1}{\frac{1+x}{1-x}+1}=x$ Hence true, QED, tick etc.	M1 A1	1.1b 2.1
(b)	$2x=\tanh\left(\ln\sqrt{2-3x}\right)\Rightarrow \tanh^{-1}(2x)=\ln\sqrt{2-3x}$	M1	3.1a
	$\dfrac{1}{2}\ln\left(\dfrac{1+2x}{1-2x}\right)=\dfrac{1}{2}\ln(2-3x)\Rightarrow\dfrac{1+2x}{1-2x}=2-3x$	M1	2.1
	$6x^{2}-9x+1=0$	A1	1.1b
	$6x^{2}-9x+1=0\Rightarrow x=\ldots$	M1	1.1b
	$x=\dfrac{9-\sqrt{57}}{12}$	A1	3.2a
		(5)	

	Alternative for first 2 marks of (b)		
	$2x = \tanh\left(\ln\sqrt{2-3x}\right) \Rightarrow 2x = \dfrac{e^{2\ln\sqrt{2-3x}} - 1}{e^{2\ln\sqrt{2-3x}} + 1}$	M1	3.1a
	$\Rightarrow \dfrac{2-3x-1}{2-3x+1} = 2x$	M1	2.1
			(10 marks)

Notes

(a)

<u>If you come across any attempts to use calculus to prove the result – send to review</u>

M1: Begins the proof by expressing tanh in terms of exponentials and forms an equation in exponentials.

The exponential form can be any of $\dfrac{\left(e^{y} - e^{-y}\right)/2}{\left(e^{y} + e^{-y}\right)/2}$, $\dfrac{e^{y} - e^{-y}}{e^{y} + e^{-y}}$, $\dfrac{e^{2y} - 1}{e^{2y} + 1}$

Allow any variables to be used **but the final answer must be in terms of *x***. Allow alternative notation for $\tanh^{-1} x$ e.g. artanh, arctanh.

A1: Correct expression for "x" in terms of exponentials

M1: Full method to make $e^{2"y"}$ the subject of the formula. This must be correct algebra so allow sign errors only.

A1*: Completes the proof by using logs correctly and reaches the printed answer with no errors. Allow e.g. $\frac{1}{2}\ln\left(\frac{x+1}{1-x}\right)$, $\frac{1}{2}\ln\frac{x+1}{1-x}$, $\frac{1}{2}\ln\left|\frac{x+1}{1-x}\right|$. Need to see $\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ as a conclusion but allow if the proof concludes that $y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ with y defined as $\tanh^{-1} x$ earlier.

B1: Correct value for k or writes $-1 < x < 1$

Way 2

M1: Starts with result, takes tanh of both sides and expresses in terms of exponentials

A1: Correct expression

M1: Eliminates exponentials and logs and simplifies

A1: Correct result (i.e. $x = x$) with conclusion

B1: Correct value for k or writes $-1 < x < 1$

(b)

M1: Adopts a correct strategy by taking $\tanh^{-1}$ of both sides

M1: Makes the link with part (a) by replacing artanh($2x$) with $\frac{1}{2}\ln\left(\frac{1+2x}{1-2x}\right)$ and demonstrates the use of the power law of logs to obtain an equation with logs removed **correctly**.

A1: Obtains the correct 3TQ

M1: Solves their 3TQ using a correct method (see General Guidance – if no working is shown (calculator) and the roots are correct for their quadratic, allow M1)

A1: Correct value with the other solution rejected (accept rejection by omission) so $x = \dfrac{9 \pm \sqrt{57}}{12}$ scores A0 unless the positive root is rejected

Alternative for first 2 marks of (b)

M1: Adopts a correct strategy by expressing tanh in terms of exponentials

M1: Demonstrates the use of the power law of logs to obtain an equation with logs removed correctly

Question	Scheme	Marks	AOs
1	$\frac{dy}{dx} = 31\cosh x - 4\cosh 2x$	B1	1.1b
	$\frac{dy}{dx} = 31\cosh x - 4\left(2\cosh^2 x - 1\right)$	M1	3.1a
	$8\cosh^2 x - 31\cosh x - 4 = 0$	A1	1.1b
	$(8\cosh x + 1)(\cosh x - 4) = 0 \Rightarrow \cosh = \ldots$	M1	1.1b
	$\cosh x = 4, \left(-\frac{1}{8}\right)$	A1	1.1b
	$\cosh x = \alpha \Rightarrow x = \ln\left(\alpha + \sqrt{\alpha^2 - 1}\right)$ or $\ln\left(\alpha + \sqrt{\alpha^2 - 1}\right)$ or $-\ln\left(\alpha + \sqrt{\alpha^2 - 1}\right)$ or $\ln\left(\alpha - \sqrt{\alpha^2 - 1}\right)$ or $\frac{e^x + e^{-x}}{2} = 4 \Rightarrow e^{2x} - 8e^x + 7 = 0 \Rightarrow e^x = \ldots \Rightarrow x = \ln(\ldots)$	M1	1.2
	$\pm\ln\left(4 + \sqrt{15}\right)$ or $\ln\left(4 \pm \sqrt{15}\right)$	A1	2.2a
		(7)	
	Alternative $\frac{dy}{dx} = 31\cosh x - 4\cosh 2x$ or $31\left(\frac{e^x + e^{-x}}{2}\right) - 4\left(\frac{e^{2x} + e^{-2x}}{2}\right)$	B1	1.1b
	Using $\cosh x = \left(\frac{e^x + e^{-x}}{2}\right)$ and $\sinh x = \left(\frac{e^x - e^{-x}}{2}\right)$ as required $\Rightarrow 31\left(\frac{e^x + e^{-x}}{2}\right) - 4\left(\frac{e^{2x} + e^{-2x}}{2}\right) = 0$ leading to $4e^{4x} - 31e^{3x} - 31e^x + 4 = 0$ o.e.	M1 A1	3.1a 1.1b
	Solves $4e^{4x} - 31e^{3x} - 31e^x + 4 = 0$ $\Rightarrow e^x = \ldots$	M1	1.1b
	$e^x = 4 \pm \sqrt{15}$ or awrt 7.87, 0.13	A1	1.1b
	$x = \ln(b)$ where b is a real exact value	M1	1.2
	$\ln\left(4 \pm \sqrt{15}\right)$	A1	2.2a
		(7)	
		(7 marks)	

Notes

B1: Correct differentiation

M1: Identifies a correct approach by using a correct identity to make progress to obtain a quadratic in cosh x

A1: Correct 3 term quadratic obtained

M1: Solves their 3TQ

A1: Correct values (may only see 4 here)

M1: Correct process to reach at least one value for x from their cosh x

Question	Scheme	Marks	AOs
2	Solves the quadratic equation for $cosh^2 x$ e.g. $\left(8\,cosh^2 x-9\right)\left(8\,cosh^2+1\right)=0 \Rightarrow cosh^2 x=\ldots$	M1	3.1a
	$cosh^2 x=\frac{9}{8}\ \left\{-\frac{1}{8}\right\}$	A1	1.1b
	$cosh\,x=\frac{3}{4}\sqrt{2} \Rightarrow x=ln\left[\frac{3}{4}\sqrt{2}+\sqrt{\left(\frac{3}{4}\sqrt{2}\right)^2-1}\right]$ **Alternatively** $cosh\,x=\frac{3}{4}\sqrt{2} \Rightarrow \frac{1}{2}(e^x+e^{-x}) \Rightarrow e^{2x}-\frac{3}{2}\sqrt{2}e^x+1=0$ $\Rightarrow e^x=\sqrt{2}$ or $\frac{\sqrt{2}}{2} \Rightarrow x=\ldots$	M1	1.1b
	$x=\pm\frac{1}{2}ln\,2$	A1	2.2a
		(4)	
			(4 marks)

Notes:

M1: Solves the quadratic equation for $cosh^2 x$by any valid means. If by calculator accept for reaching the positive value for $cosh^2 x$ (negative may be omitted or incorrect) but do not allow for going directly to a value for $cosh\,x$ Alternatively score a correct process leading to a value for sinh $2x$ or its square (Alt 1) or use of correct exponential form for cosh x to form and expand to an equation in e^{4x} and e^{2x} (Alt 2)

A1: Correct value for $cosh^2 x$ (ignore negative or incorrect extra roots.). In Alt 1 score for a correct value for $\sinh^2 2x$ or sinh $2x$. In Alt 2 score for a correct simplified equation in e^{4x}.

M1: For a correct method to achieve at least one value for x (from $\cosh^2 x$). In the main scheme or Alt 1, takes positive square root (if appropriate) and uses the correct formula for arcosh x or arsinh x to find a value for x. (No need to see negative square root rejected.) In Alt 2 it is for solving the quadratic in e^{4x} and proceeding to find a value for x.

Alternatively uses the exponential definition for cosh x, forms and solves a quadratic for e^x leading to a value for x

A1: Deduces (both) the correct values for x and no others. Must be in the form specified.

SC Allow M0A0M1A1 for cases where a calculator was used to get the value for cosh x with no evidence if a correct method for find both values is shown.